Chua's and Feigenbaum Non-Linear Circuits

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(Dated: 22 November,2017)

Abstract

A nonlinear system is a system in which the change of the output is not proportional to the change of the input. Nonlinear problems are of interest to engineers, physicists, mathematicians, and many other scientists because most systems are inherently nonlinear in nature. In this experiment, we construct two such circuits which demonstrates non-linear and chaotic phenomena. The Chua's circuit is a simple electronic network which exhibits a variety of bifurcation and attractors. The circuit consists of two capacitors, an inductor, a linear resistor and a nonlinear resistor. The Feigenbaum circuit is used to simulate the iterated logistic map and to study the bifurcation plot.

CONTENTS

Chaos: When the present determines the future, but the approximate present does not approximately determine the future.

Edward Lorenz

NON-LINEAR DYNAMICS AND CHAOS

A nonlinear system is a system in which the change of the output is not proportional to the change of the input. Nonlinear dynamics, also popularly known as chaos (see Chaos Theory) is the study of systems governed by equations in which a small change in one variable can induce a large systematic change in the final state of the system. Unlike a linear system, in which a small change in one variable produces a small and easily quantifiable systematic change, a nonlinear system exhibits a sensitive dependence on initial conditions: small or virtually unmeasurable differences in initial conditions can lead to wildly differing outcomes. This sensitive dependence is sometimes referred to as the "*butterfly effect*," the assertion that the beating of a butterfly's wings in Brazil can eventually cause a hurricane in Texas. This happens even though these systems are deterministic, meaning that their future behavior is fully determined by their initial conditions, with no random elements involved.[4] In other words, the deterministic nature of these systems does not make them predictable. This behavior is known as deterministic chaos, or simply chaos.

There are two main types of dynamical systems: **differential equations** and **iterated maps**. Differential equations describe the evolution of systems in continuous time, whereas iterated maps arise in problems where time is discrete.
Any general system can be described in terms of differential equations as

$$\dot{x}_i = f_i(x_1, \ldots, x_n)$$

where i ranges from 1 to n, where n is the number of variables in the system.

Iterated systems can be described by recursion relations of the form

$$x_{n+1} = f(x_n)$$

The rule to obtain x_{n+1} from $f(x_n)$ is known as a one-dimensional map. The sequence $x_0, x_1, x_2, \ldots$ is called the orbit starting from x_0.

AIM

In this experiment, we demonstrate the two types of nonlinear systems with the help of Chua's circuit (Differential equations system) and Feigenbaum circuit (Iterated system).

CHUA'S CIRCUIT

In this experiment we have developed some intuition into chaotic phenomenon using a circuit called Chua circuit.The criterion for choosing Chua's circuit is its simplicity, though

simple, it exhibits a variety of chaotic phenomena exhibited by other complex circuits, which makes it a popular circuit. There are two types of chaotic systems, autonomous and non-autonomous. Chua's circuit is an autonomous system because there is no external signal injected into the system.
An autonomous circuit made from standard components (resistors, capacitors, inductors) must satisfy three criteria before it can display chaotic behaviour. It must contain:

1. one or more nonlinear elements

2. one or more locally active resistors

3. three or more energy-storage elements.

Chua's circuit is the simplest electronic circuit meeting these criteria. Chua's circuit as shown in Fig.1 consists of two capacitors,an inductor and a nonlinear resistor NR (a pair of negative resistors.)

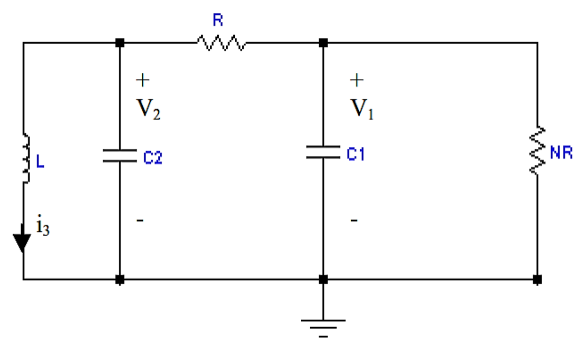

FIG. 1. The unfolded Chua's circuit.

Applying KCL and KVl to the circuit,we get

$$C_1 \frac{dv_1}{dt} = \frac{1}{R}(v_1 - v_2) - f(v_1) \tag{1}$$

$$C_2 \frac{dv_2}{dt} = \frac{1}{R}(v_1 - v_2) + i_3 \tag{2}$$

$$L \frac{di_3}{dt} = -v_2 \tag{3}$$

where the piecewise linear Chua's function is given by

$$f(v_R) = m_0 v_R + \frac{1}{2}(m_1 - m_0)\left[|v_R + B_p| - |v_R - B_p|\right] \tag{4}$$

where v_R is the voltage across the NR and $f(v_R)$ is the current passing through it. B_p is the "breaking point" of the function.

$$m_1 = \frac{-R_2}{R_1 R_3} - \frac{R_5}{R_4 R_6}, m_0 = \frac{-R_2}{R_1 R_3} + \frac{1}{R_4}, B_{p1} = \frac{R_3}{R_2 + R_3} E_{sat}, B_{p2} = \frac{R_6}{R_5 + R_6} E_{sat}$$

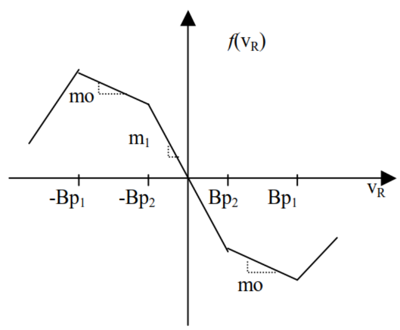

FIG. 2. Chua's nonlinear function.

Negative Resistance(NR)

Negative resistance (NR) is a property of some electrical circuits and devices in which an increase in voltage across the device's terminals results in a decrease in electric current through it.This is in contrast to an ordinary resistor in which an increase of applied voltage causes a proportional increase in current due to Ohm's law, resulting in a positive resistance.While a positive resistance consumes power from current passing through it, a negative resistance produces power. Under certain conditions it can increase the power of an electrical signal, amplifying it.

In a nonlinear device, two types of resistance can be defined:

1. 'Static' or 'absolute resistance': the ratio of voltage to current v/i 2. Differential resistance, the ratio of a change in voltage to the resulting change in current $\triangle v/\triangle i$

The term negative resistance means **negative differential resistance (NDR)**, $\boldsymbol{\triangle v/\triangle i < 0}$.

The NR is constructed using two resistance in parallel to each other. The negative resistances are constructed by using the TL082 Op-amp with suitable values of passive resistors so as to demonstrate the chaotic phenomenon.

Experimental Verification of i-v characteristics

The current through the NR was measured for a range of values of applied voltage and the I-V characteristic plot was obtained. It can be seen that the slope of the graph is negative within the operational region (withing the breaking points) as expected because it is a negative resistance I-V plot. From the graph 3, we obtained

$$Bp_1 = \frac{11.0123 + 10.499}{2} = 10.756$$

$$Bp_2 = \frac{1.984 + 1.942}{2} = 1.963$$

$$m_1 = -0.00059 \ ; \ m_0 = -\frac{0.000265 + 0.000268}{2} = -0.000267$$

Thus,the graph is as predicted by theory.

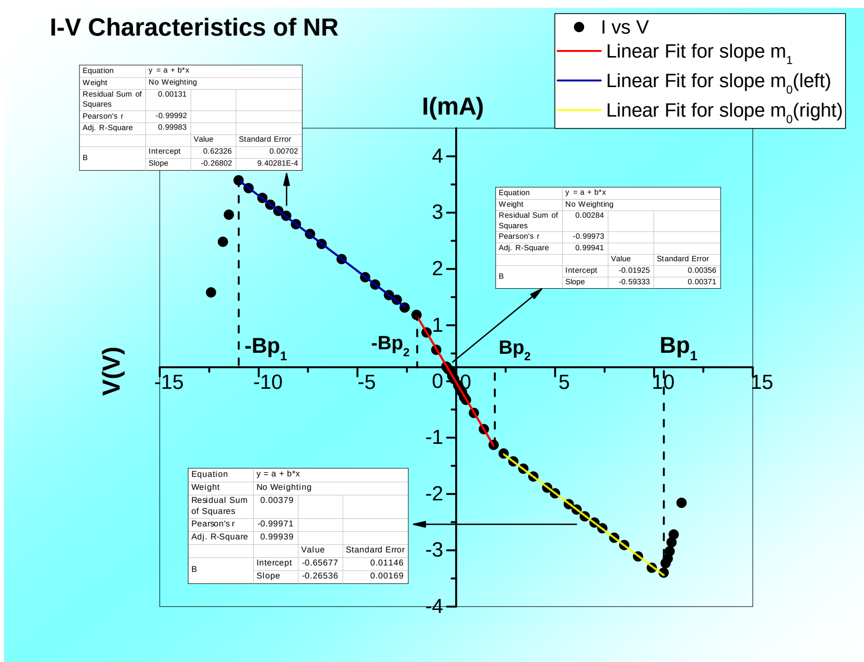

FIG. 3. I-V characteristics of NR.

Circuit Implementation

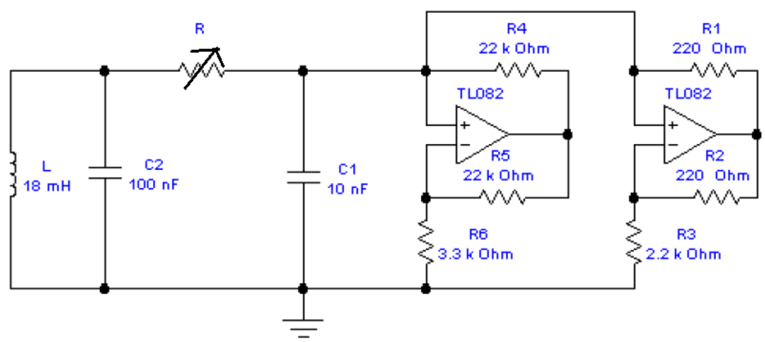

FIG. 4. The realization of the Chua's circuit.

The negative resistance was constructed by using the TL082 Op-amp with suitable values of passive resistors so as to demonstrate the chaotic phenomenon. Voltage across the capacitors C1 and C2 are measured to be v_1 and v_2 and fed to the oscilloscope. The variable resistance R is varied and the waveforms of $v_{C1}(t)$ and $v_{C2}(t)$ was observed.Fig.5 and 6

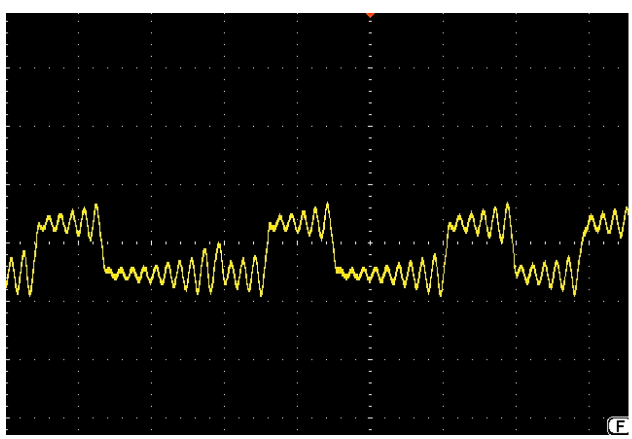

FIG. 5. Waveform $V_{C1}(t)$

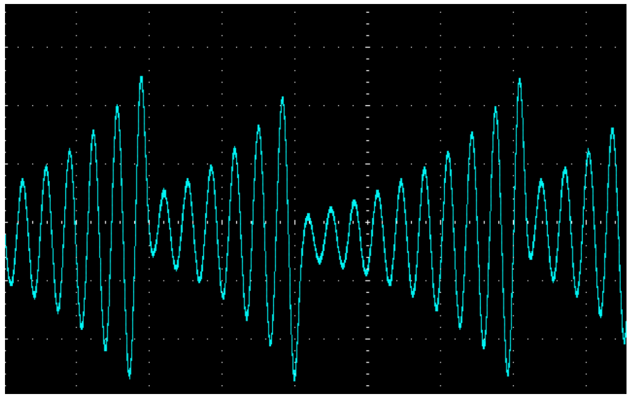

FIG. 6. Waveform $V_{C2}(t)$

R bifurcation sequence

Chua's circuit exhibits a sequence of bifurcations from dc equilibrium through a Hopf bifurcation and period-doubling sequence to a Rössler-type attractor and finally the Double Scroll strange attractor. A two-dimensional project of the attractor (Poincaré Section) is obtained by connecting v_{c1} and v_{c2} to the X and Y channels, respectively of an XY-mode oscilloscope. Fig.7

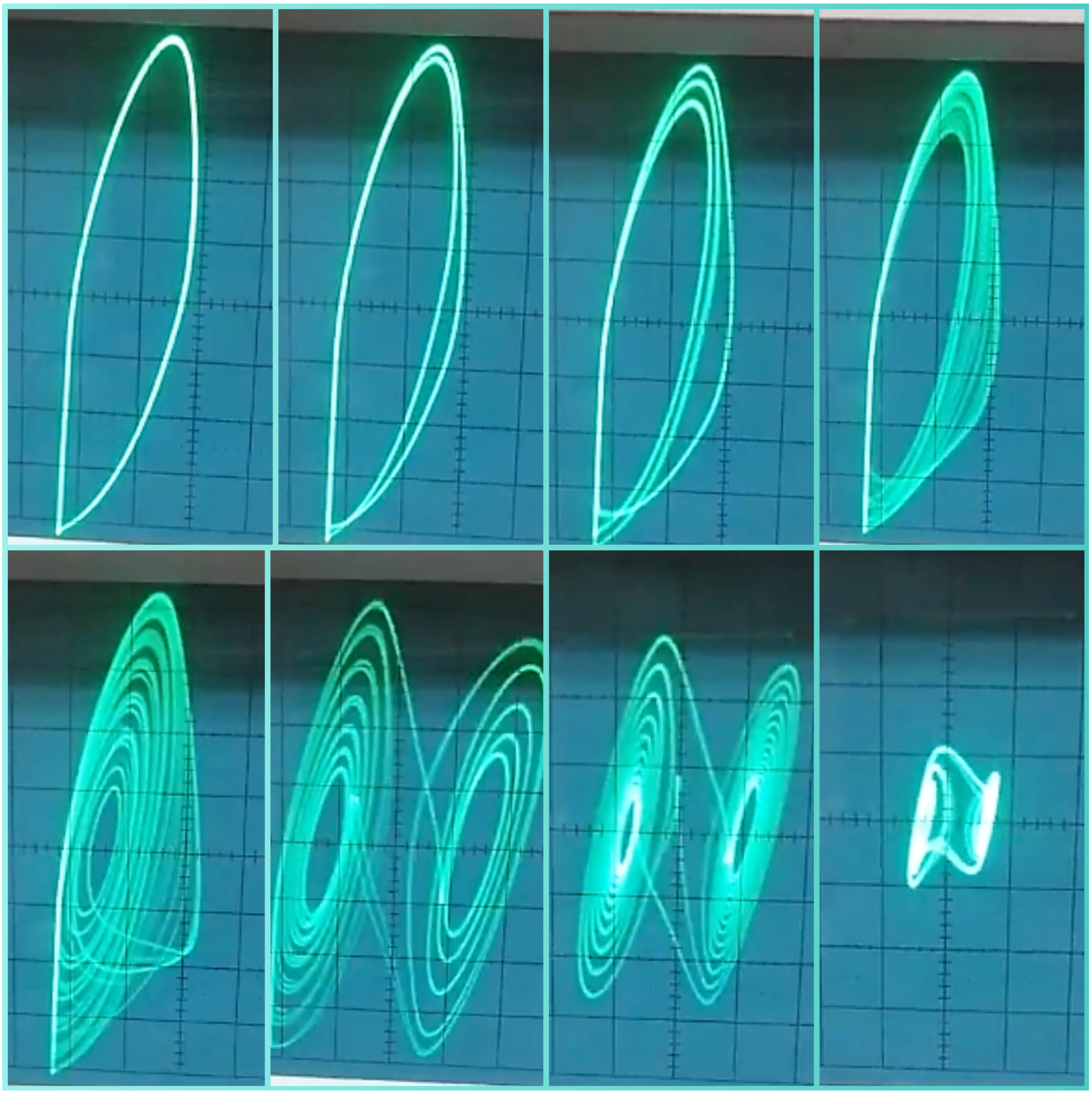

FIG. 7. Typical R bifurcation sequence in Chua's circuit. Top row: Period cycles of 1,2,3 and more are formed. Bottom row: Rössler-type attractor is formed. The attracted just splits up intqo two at $R \approx 1.8$ kOhm. Then the double scroll attractor is formed. The attractor finally dies off with the limit cycle in the end.

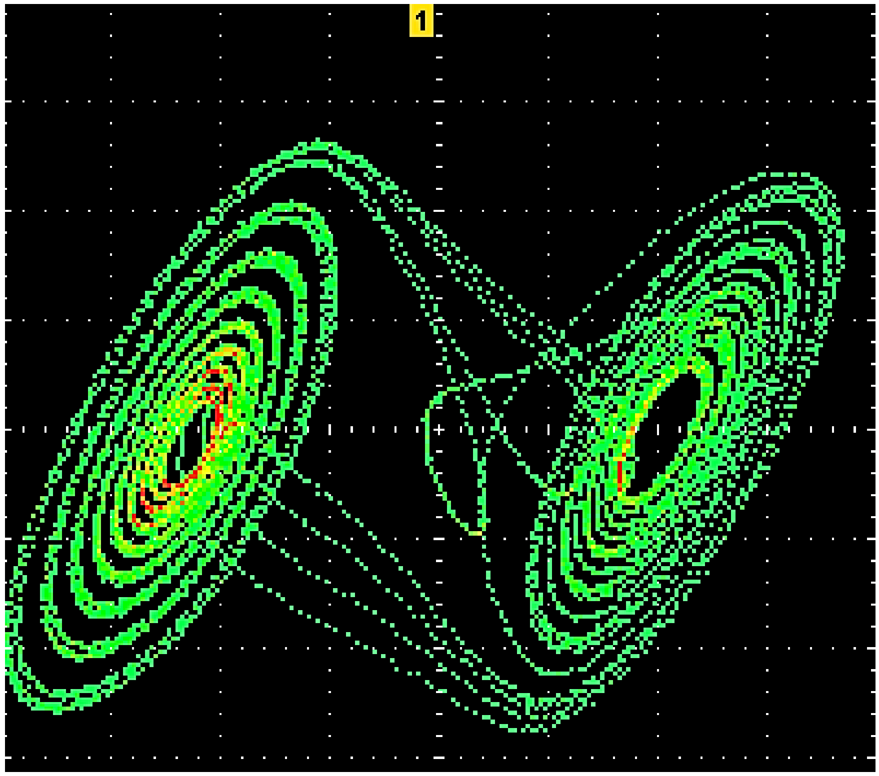

FIG. 8. The double scroll strange attractor formed in the oscilloscope

Numerical Simulations

The dimensionless chua's differential equations were solved using the Fourth Order Runge-Kutta method for 3-dimension. The plots obtained from the circuit during the experiment were reproduced and were found to be of similar nature.

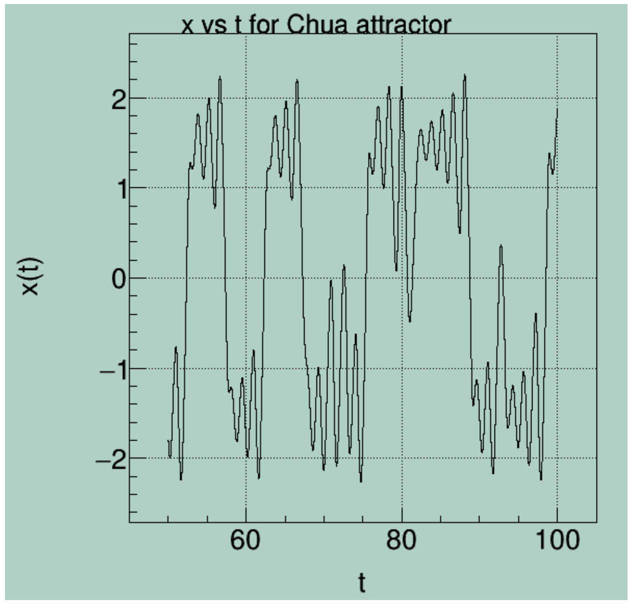

FIG. 9. Plot of x(t) corresponding to waveform $V_{c1}(t)$

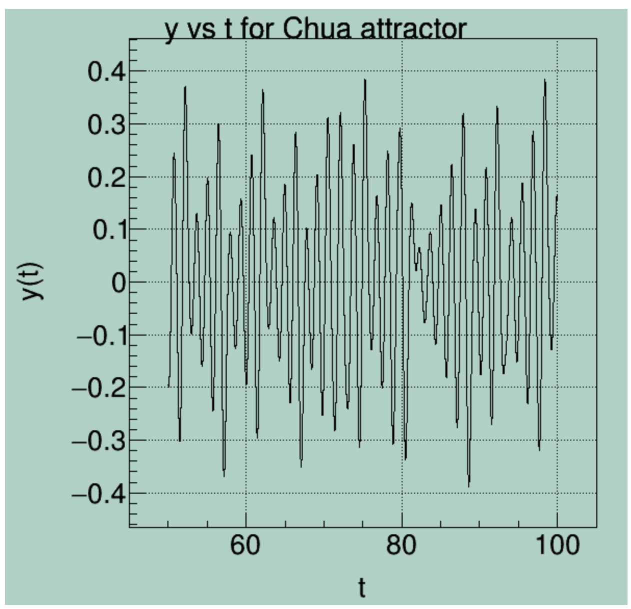

FIG. 10. Plot of y(t) corresponding to waveform $V_{c2}(t)$

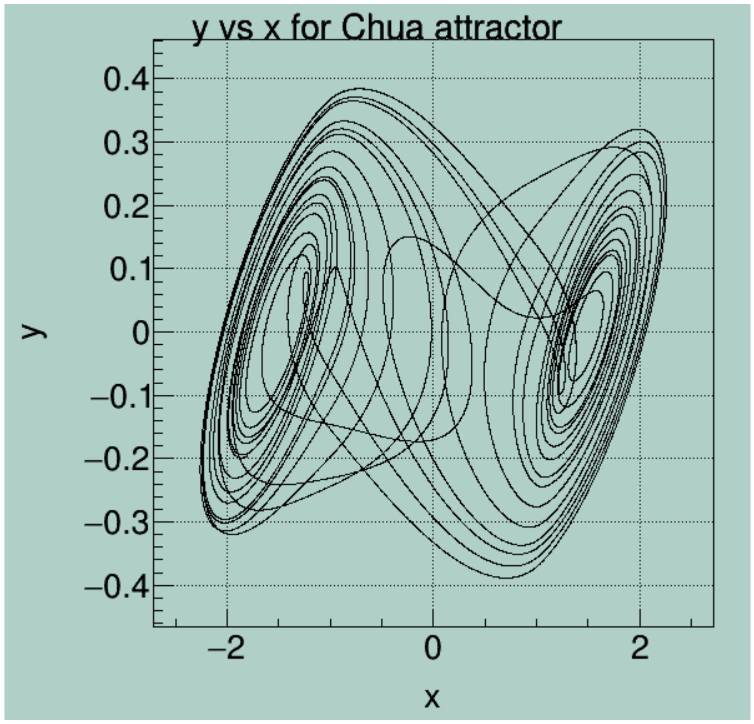

FIG. 11. The double scroll strange attractor obtained numerically.

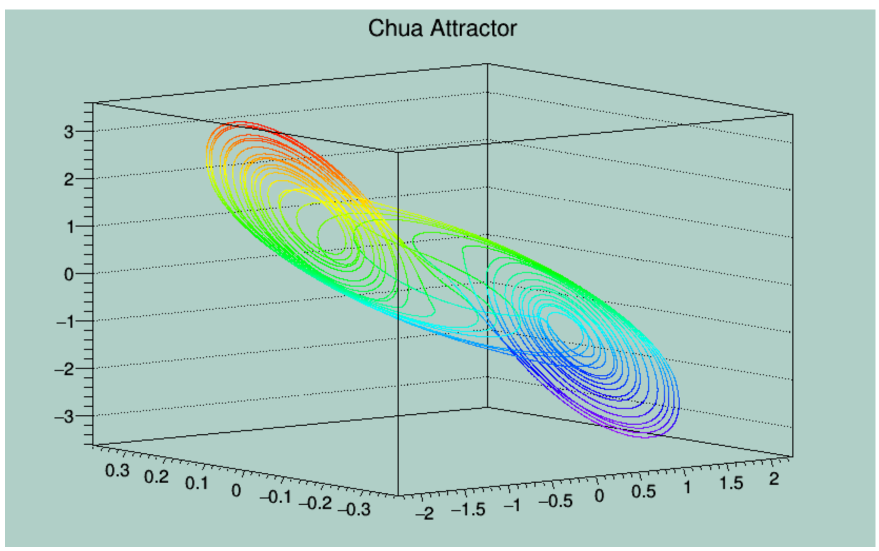

FIG. 12. The Chua's attractor in 3D

FEIGENBAUM CIRCUIT

The Feigenbaum circuit is a standard example of introducing the concept of deterministic chaos in terms of iterated quadratic function map dependent on parameter r.

$$x_{n+1} = rx_n(1 - x_n), (n = 0, 1, 2, \dots, 0 < r < 4, 0 < x_0 < 1) \quad (5)$$

Logistic Map

We have $x_{n+1} = rx_n(1 - x_n)$. The fixed points satisfy the relation

$$x* = f(x*) = rx * (1 - x*)$$

Hence,we get the solutions as $x* = 0$ and $x* = 1 - \frac{1}{r}$
The stability of fixed points depends on its derivative. Therefore,

$$f'(x*) = r - 2rx*$$

Since, $f'(0) = r$, the origin $x* = 0$ is stable for $r < 1$.
Derivative value at the other fixed point is given by:

$$f'(x*) = r - 2r(1 - \frac{1}{r}), \text{ stable for } 1 < r < 3$$

The fixed point upto $r = 3$ then bifurcates into an infinte number of times (at critical values of r) and exhibit periods of 2,4,8,16,... and eventually becomes chaotic at a value $r_\infty \approx 3.57$

The circuit made here is used to demonstrate this phenomenon.Fig.13

Circuit Implementation

The idea of the circuit is to represent the quantity x by a voltage ($v = x \times 10\ V$) and form the product $rx(1 - x)$ with two analog transconductance multipliers. As shown in the circuit diagram, we have used AD663 chips as a multiplier.
The output voltage is then stored for a short period by a sample-and-hold circuit before being passed back by the multipliers for the next iteration. We need to use two sample-and-hold circuits. The second one holds the result of the previous iteration while the new result settles and is sampled by the first sample-and-hold.
Our goal is to manually vary r from $0 < r < 4$ so that we can get the bifurcation plot of x(Output) vs r.

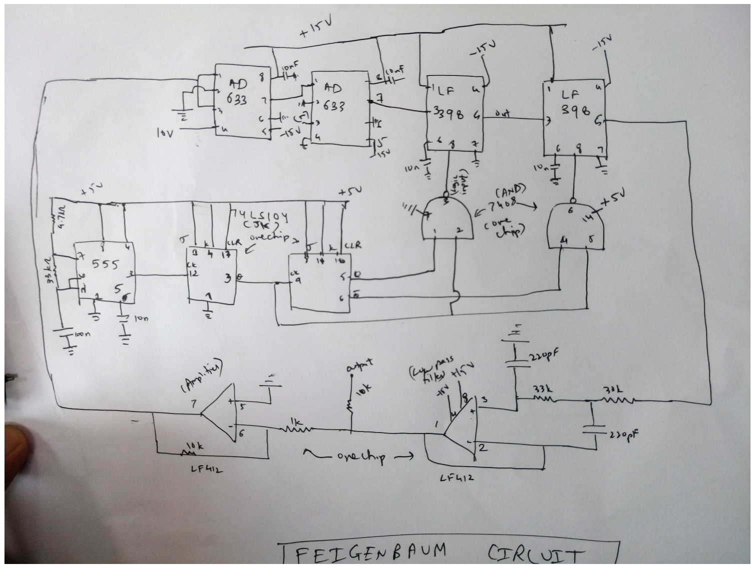

FIG. 13. Feigenbaum Circuit.

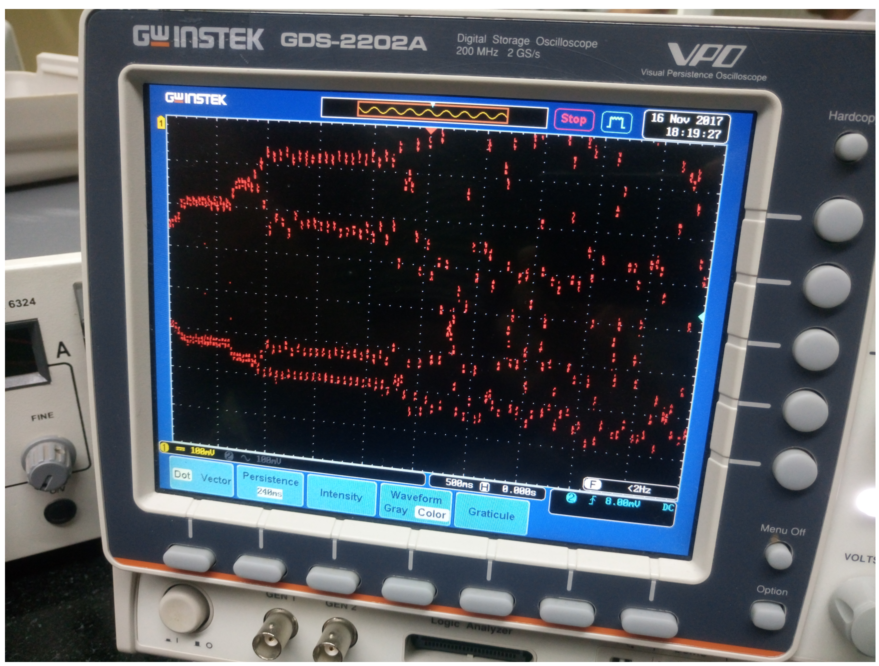

FIG. 14. Bifurcation plot of x vs r obtained in oscilloscope.

Feigenbaum constant

Feigenbaum constants δ_n are defined as

$$\delta_n = \frac{r_n - r_{n-1}}{r_{n+1} - r_n} \tag{6}$$

In this experiment, we were able to calculate only δ_1 because the higher bifurcations could not be easily resolved. $\delta_1 = \frac{3.45-3}{3.55-3.45} = 4.5$

Note: The successive bifurcations in logistic map come faster and faster. Hence, the distance between successive transitions shrinks by a constant factor:

$$\delta = \lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4.669... \tag{7}$$

Bifurcation plot by numerical analysis

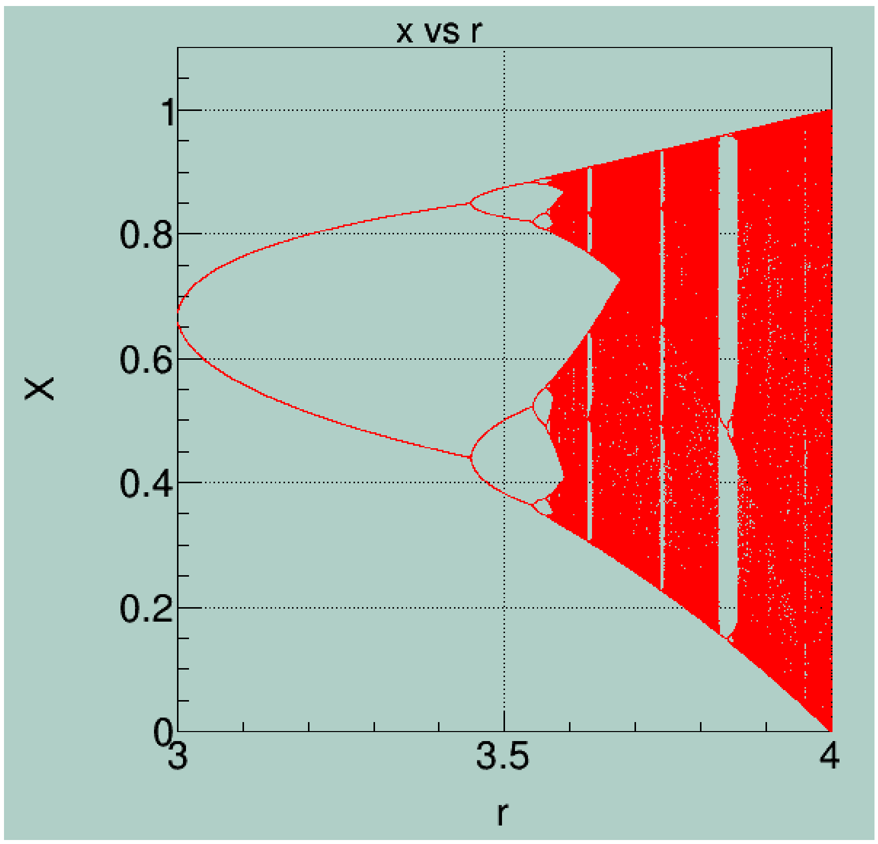

FIG. 15. x vs r bifurcation plot.

CONCLUSION

Chua's Circuit:

1. The IV characteristics of Negative resistance was obtained.

2. The Double scroll strange attractor was obtained by constructing the Chua's circuit. The Double Scroll attractor forms at $R \approx 1.8$ kOhm.

Feigenbaum Circuit:

1. The Feigenbaum circuit was constructed using multipliers and Sample-and-Hold circuits. It is used to simulate the Logistic Map.

2. The bifurcation plot of the Logistic Map was obtained and it was similar to the one obtained by numerical analysis.

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[2] Strogatz, S. H. (2014). Nonlinear dynamics and chaos: with applica- tions to physics, biology, chemistry, and engineering. Westview press.

[3] Implementation of the Chua's circuit and its applications. Mahesh Mulukutla and Cherif Aissi. University of Louisiana at Lafayette, College of Engineering Lafayette, LA 70504, USA

[4] Simple experiments in chaotic dynamics, Keith Briggs.