Non-Linear Circuits and Applications.

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Abstract

A nonlinear system is a system in which the change of the output is not proportional to the change of the input. Nonlinear problems are of interest to engineers, physicists, mathematicians, and many other scientists because most systems are inherently nonlinear in nature. In this experiment, We study the Lorenz system of chaotic strange attractor. We implement the Lorenz circuit numerically and experimentally to obtain various plots amongst it's variables. Further, We study about the Self-Synchronization property of the Lorenz system and learn about it's application in Chaotic Signal Masking. Also, We implement the PWL circuits for the square and cubic functions.

CONTENTS

Chaos: When the present determines the future, but the approximate present does not approximately determine the future.

Edward Lorenz

NON-LINEAR DYNAMICS AND CHAOS

A nonlinear system is a system in which the change of the output is not proportional to the change of the input. Nonlinear dynamics, also popularly known as chaos (see Chaos Theory) is the study of systems governed by equations in which a small change in one variable can induce a large systematic change in the final state of the system. Unlike a linear system, in which a small change in one variable produces a small and easily quantifiable systematic change, a nonlinear system exhibits a sensitive dependence on initial conditions: small or virtually unmeasurable differences in initial conditions can lead to wildly differing outcomes. This sensitive dependence is sometimes referred to as the "*butterfly effect*," the assertion that the beating of a butterfly's wings in Brazil can eventually cause a hurricane in Texas. This happens even though these systems are deterministic, meaning that their future behavior is fully determined by their initial conditions, with no random elements involved.[4] In other words, the deterministic nature of these systems does not make them predictable. This behavior is known as deterministic chaos, or simply chaos.

There are two main types of dynamical systems: **differential equations** and **iterated maps**. Differential equations describe the evolution of systems in continuous time, whereas iterated maps arise in problems where time is discrete.
Any general system can be described in terms of differential equations as

$$\dot{x}_i = f_i(x_1, \ldots, x_n)$$

where i ranges from 1 to n, where n is the number of variables in the system.

Iterated systems can be described by recursion relations of the form

$$x_{n+1} = f(x_n)$$

The rule to obtain x_{n+1} from $f(x_n)$ is known as a one-dimensional map. The sequence $x_0, x_1, x_2, \ldots$ is called the orbit starting from x_0.

Terminologies

Fractal A fractal is a never-ending pattern. Fractals are infinitely complex patterns that are self-similar across different scales. They are created by repeating a simple process over and over in an ongoing feedback loop. Driven by recursion, fractals are images of dynamic systems – the pictures of Chaos. Geometrically, they exist in between our familiar dimensions. Fractals can have fractional dimensions.

Example: **The Sierpinski triangle** is a fractal with the overall shape of an equilateral triangle, subdivided recursively into smaller equilateral triangles. Originally constructed as a curve, this is one of the basic examples of self-similar sets, i.e., it is a mathematically generated pattern that can be reproducible at any magnification or reduction.

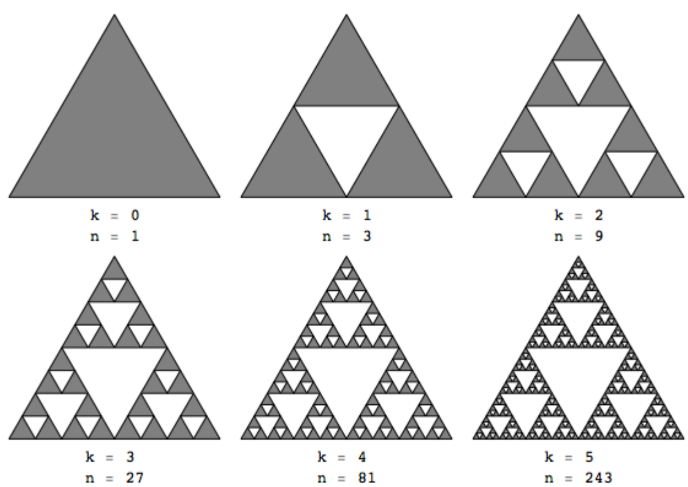


FIG. 1. Example of a fractal: The Sierpinski triangle

Attractor An attractor is a set of states (points in the phase space), invariant under the dynamics, towards which neighboring states in a given basin of attraction asymptotically approach in the course of dynamic evolution. An attractor is defined as the smallest unit which cannot be itself decomposed into two or more attractors with distinct basins of attraction. This restriction is necessary since a dynamical system may have multiple attractors, each with its own basin of attraction. There are many types of attractors as shown in the Fig.2. We are interested to study about the strange attractor.

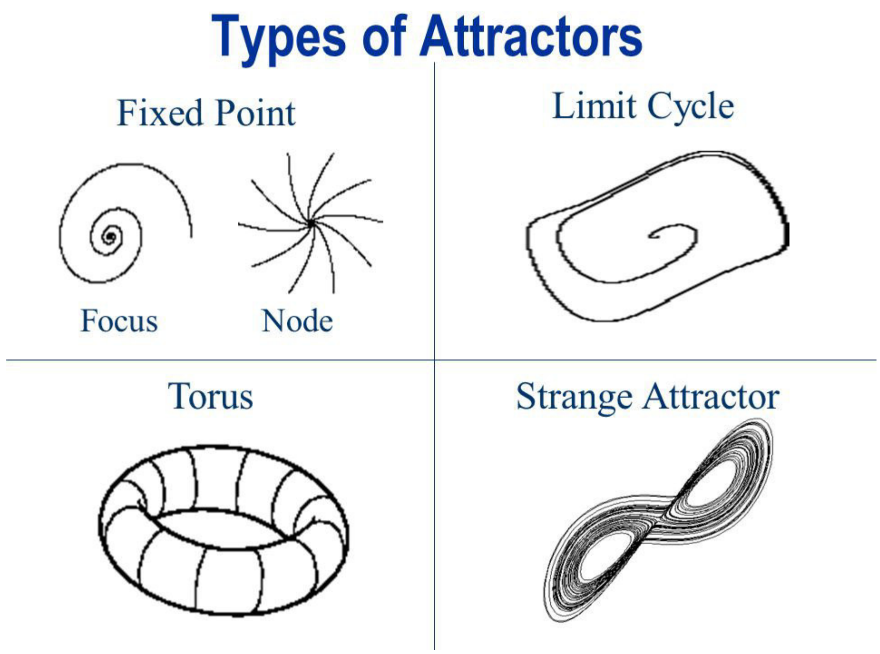


FIG. 2. Types of attractors

Basin of Attraction An attractor's basin of attraction is the region of the phase space, over which iterations are defined, such that any point (any initial condition) in that region will eventually be iterated into the attractor.

Strange Attractor An attractor is called strange if it has a fractal structure. This is often the case when the dynamics on it are chaotic. If a strange attractor is chaotic, exhibiting sensitive dependence on initial conditions, then any two arbitrarily close alternative initial points on the attractor, after any of various numbers of iterations, will lead to points that are arbitrarily far apart (subject to the confines of the attractor). Thus a dynamic system with a chaotic attractor is locally unstable yet globally stable: once some sequences have entered the attractor, nearby points diverge from one another but never depart from the attractor.Examples of strange attractors include the double-scroll attractor, Rössler attractor and the Lorenz attractor.

LORENZ SYSTEM

In 1963, Edward Lorenz (1917-2008), studied convection in the Earth's atmosphere. As the Navier-Stokes equations that describe fluid dynamics are very difficult to solve, he simplified them drastically. The model he obtained probably has little to do with what really happens in the atmosphere. It is a toy-model, but Lorenz soon realized that it as very interesting in a mathematical sense. The Lorenz system is a system of ordinary differential equations. There are only three parameters in the model so that each point (x,y,z) symbolizes a state of the atmosphere. The Lorenz system is described by the Lorenz equations:

$$\begin{aligned}\dot{x} &= \alpha(y-x)\\ \dot{y} &= \rho x - xz - y\\ \dot{z} &= xy - \beta z\end{aligned}$$

We choose the following parameter values in order to obtain a chaotic strange attractor:

$$\alpha = 10\,,\ \rho = 28\,,\ \beta = \frac{8}{3}$$

Why is it an attractor? The Lorenz system is *dissipative*: meaning volumes in phase space contract under the flow.
Consider the Lorentz system to be represented by three-dimensional equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. We consider an arbitrary closed surface $S(t)$ for the volume $V(t)$ in the phase space. Surface S can be thought of as initial conditions for the trajectories. These trajectories evolve into a new surface $S(t+dt)$ with a volume of $V(t+dt)$

It can easily be shown that

$$\dot{V} = \oint_V \nabla.\mathbf{f}\, dV$$

For the Lorenz system, we get :

$$\begin{aligned}\nabla.\mathbf{f} &= \frac{\partial}{\partial x}[\alpha(y-x)] + \frac{\partial}{\partial y}[\rho x - xz - y] + \frac{\partial}{\partial y}[xy - \beta z]\\ &= -\sigma - 1 - \beta < 0\end{aligned}$$

Thus, we get

$$\dot{V} = -(\sigma + 1 + b)V$$

The solution is given by:

$$V(t) = V(0)e^{-(\sigma+1+b)t}$$

Thus, we can see that the volumes in phase space shrink exponentially fast.

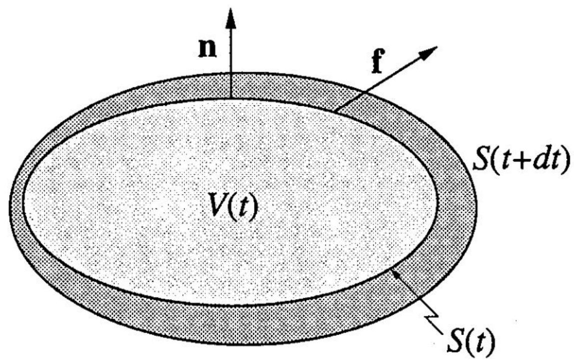


FIG. 3. Volume Element of Phase Space.

NUMERICAL SIMULATIONS OF THE LORENZ ATTRACTOR

Plots

We have solved the Lorenz system of non-linear equation of 3 variables by using **Fourth Order Runge-Kutta** (RK4) method. The following plots were obtained.

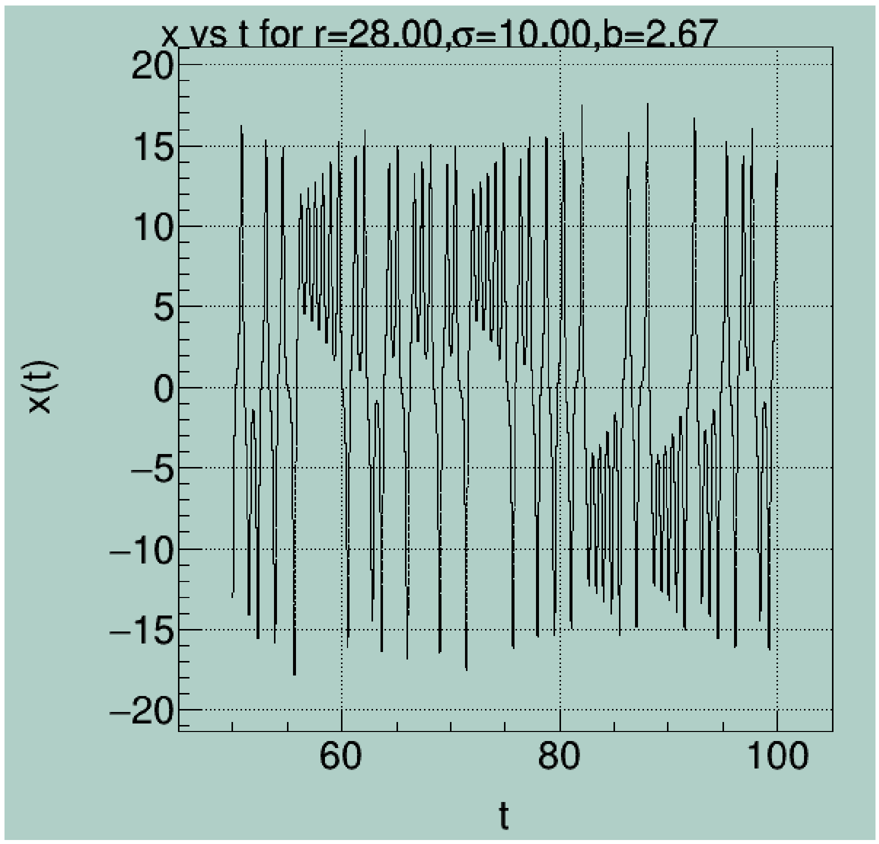


FIG. 4. Plot of x vs t.

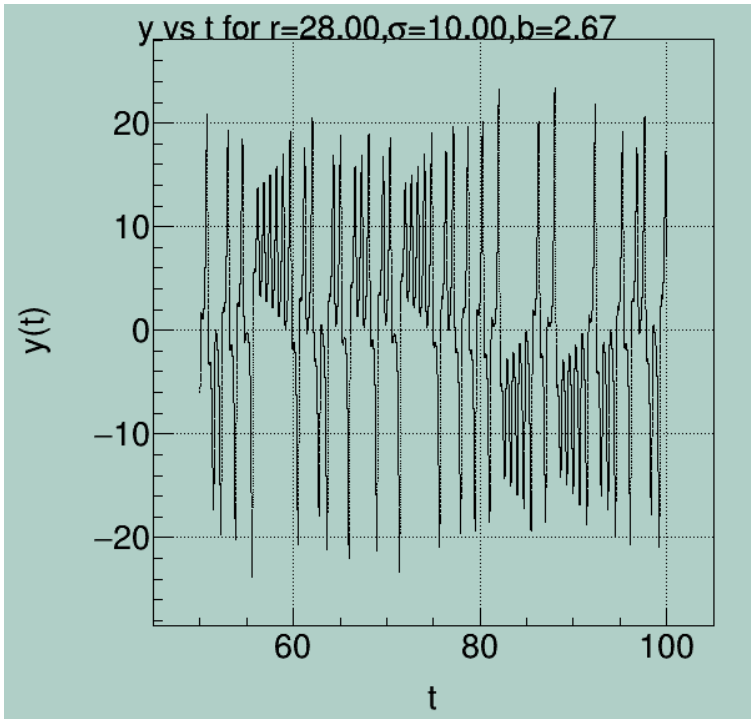


FIG. 5. Plot of y vs t.

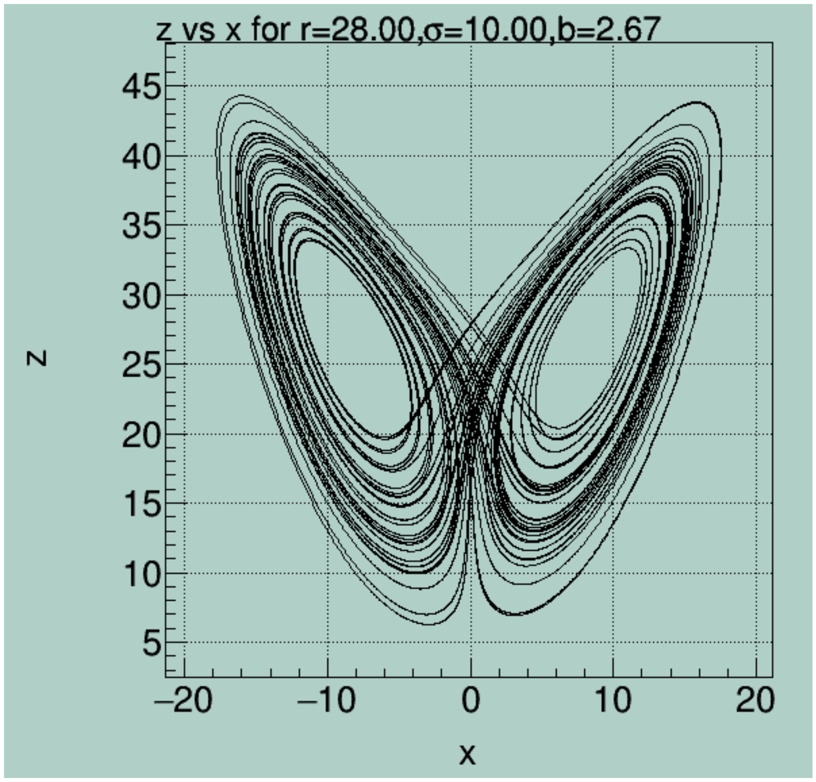


FIG. 6. Plot of z vs x which typically looks like the wings of a butterfly.

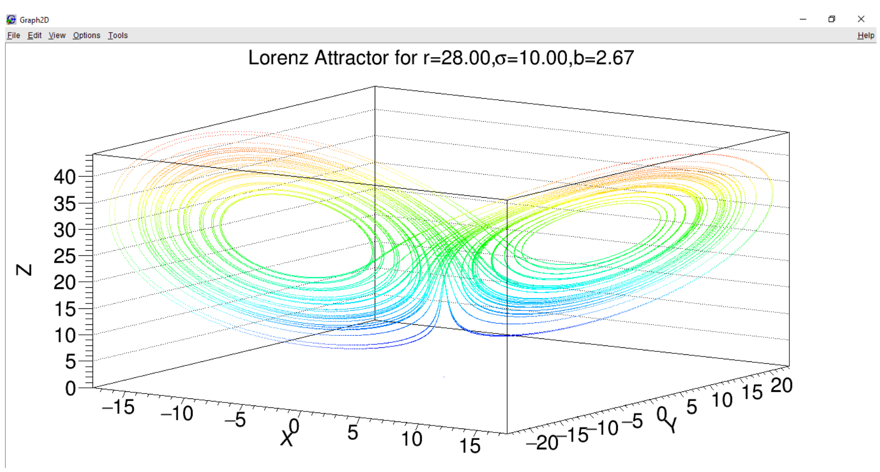


FIG. 7. A 3D view of the Lorenz Attractor.

Correlation dimension

In chaos theory, the correlation dimension is a measure of the dimensionality of the space occupied by a set of random points, often referred to as a type of fractal dimension. (See Reference[1])

The correlation dimension can be found using the formula:

$$C(\epsilon) \approx \epsilon^d \text{ where d is the dimension of the system}$$

which can be written as:

$$logC(\epsilon) \approx d\ log\epsilon$$

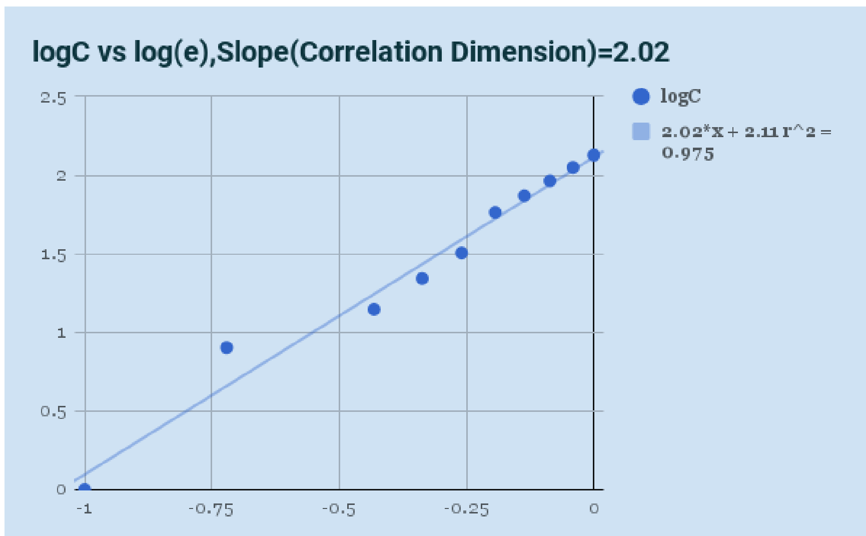


FIG. 8. Finding correlation dimension of the Lorenz attractor. The lorenz attrator is almost flat. And thus has a dimension=2.02

CIRCUIT

The circuital equations associated to the implementation of Lorenz's circuit are the following:

$$C_1R_5\frac{dX}{dt} = -X - \frac{R_4}{R_1}X + \frac{R_4}{R_2}X + \frac{R_4}{R_3}Y$$
$$C_2R_{11}\frac{dY}{dt} = -Y - \frac{R_{10}}{R_7}XZ + \frac{R_{10}}{R_8}X$$
$$C_3R_{17}\frac{dZ}{dt} = -Z - \frac{R_{16}}{R_{13}}Z + \frac{R_{16}}{R_{14}}XY$$

where

$$x = \frac{X}{k_1}\ ,\ y = \frac{Y}{k_2}\ ,\ z = \frac{Y}{k_3}$$

Here, k_1, k_2, k_3 are appropriate scaling constants for the circuit.

We have taken

$$k_1 = \frac{1}{10}\ ,\ k_2 = \frac{1}{10}\ ,\ k_3 = \frac{1}{30}$$

Thus the rescaled equivalent system becomes:

$$\dot{X} = \alpha(Y - X)$$
$$\dot{Y} = \rho X - 30XZ - Y$$
$$\dot{Z} = \frac{100}{30}XY - \beta Z$$

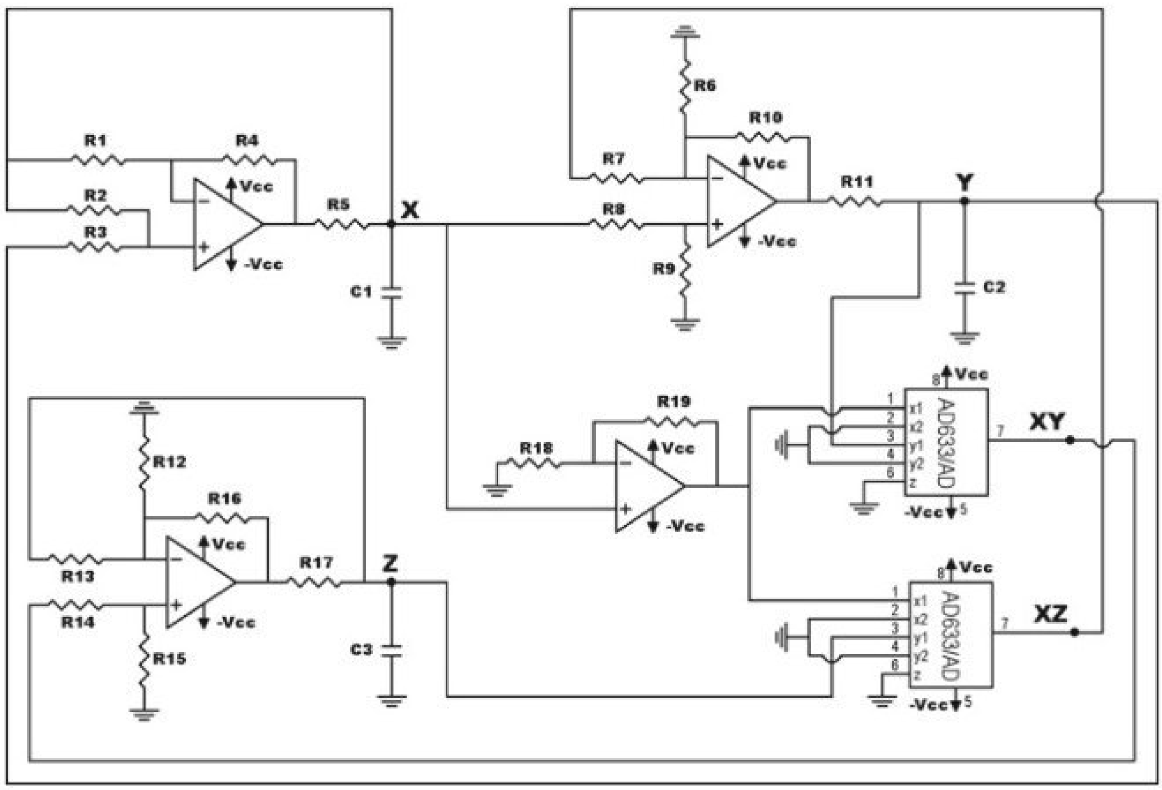


Fig. Circuital implementation Components: $R_1 = 10\ \text{k}\Omega$, $R_2 = 100\ \text{k}\Omega$, $R_3 = 10\ \text{k}\Omega$, $R_4 = 100\ \text{k}\Omega$, $R_5 = 1\ \text{k}\Omega$, $R_6 = 5.6\ \text{k}\Omega$, $R_7 = 3.3\ \text{k}\Omega$, $R_8 = 3.6\ \text{k}\Omega$, $R_9 = 3.19\ \text{k}\Omega$, $R_{10} = 100\ \text{k}\Omega$, $R_{11} = 1\ \text{k}\Omega$, $R_{12} = 3.3\ \text{k}\Omega$, $R_{13} = 37.5\ \text{k}\Omega$, $R_{14} = 3.3\ \text{k}\Omega$, $R_{15} = 3.74\ \text{k}\Omega$, $R_{16} = 100\ \text{k}\Omega$, $R_{17} = 1\ \text{k}\Omega$, $R_{18} = 1\ \text{k}\Omega$, $R_{19} = 9\ \text{k}\Omega$, $C_1 = 200$ nF, $C_2 = 200$ nF, $C_3 = 200$ nF, $V_{cc} = 9$ V

FIG. 9. Circuital implementation of Lorenz equations

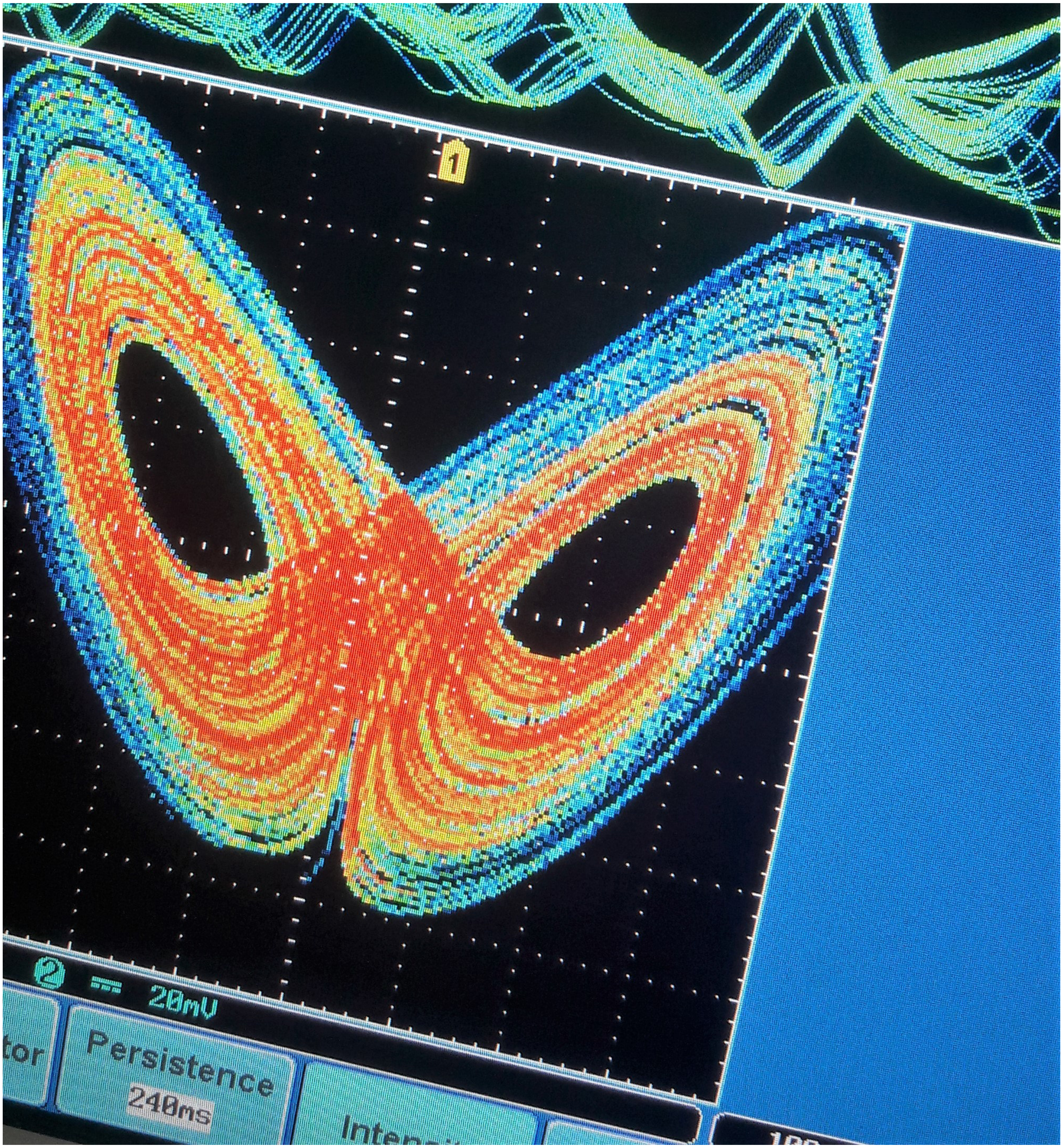


FIG. 10. A typical Lorenz circuit butterfly wings curve obtained in the oscilloscope.

PARAMETERIZATION

The circuital equations associated to the implementation of Lorenz's circuit are the following:

$$
\begin{aligned}
C_1R_5\frac{dX}{dt} &= -X - \frac{R_4}{R_1}X + \frac{R_4}{R_2}X + \frac{R_4}{R_3}Y \\
C_2R_{11}\frac{dY}{dt} &= -Y - \frac{R_{10}}{R_7}XZ + \frac{R_{10}}{R_8}X \\
C_3R_{17}\frac{dZ}{dt} &= -Z - \frac{R_{16}}{R_{13}}Z + \frac{R_{16}}{R_{14}}XY
\end{aligned}
$$

where

$$x = \frac{X}{k_1}\,,\; y = \frac{Y}{k_2}\,,\; z = \frac{Y}{k_3}$$

Here, k_1, k_2, k_3 are appropriate scaling constants for the circuit. We have taken

$$k_1 = \frac{1}{10}\,,\; k_2 = \frac{1}{10}\,,\; k_3 = \frac{1}{30}$$

Thus the rescaled equivalent system becomes:

$$
\begin{aligned}
\dot{X} &= \alpha(Y - X) \\
\dot{Y} &= \rho X - 30XZ - Y \\
\dot{Z} &= \frac{100}{30}XY - \beta Z
\end{aligned}
$$

By changing the resistors R_8 and R_{13} we effectively change the parameters ρ and β respectively. We then observe the evolution of the curve under such parameterization of these two resistors.

Parameterization on R_8 (ρ)

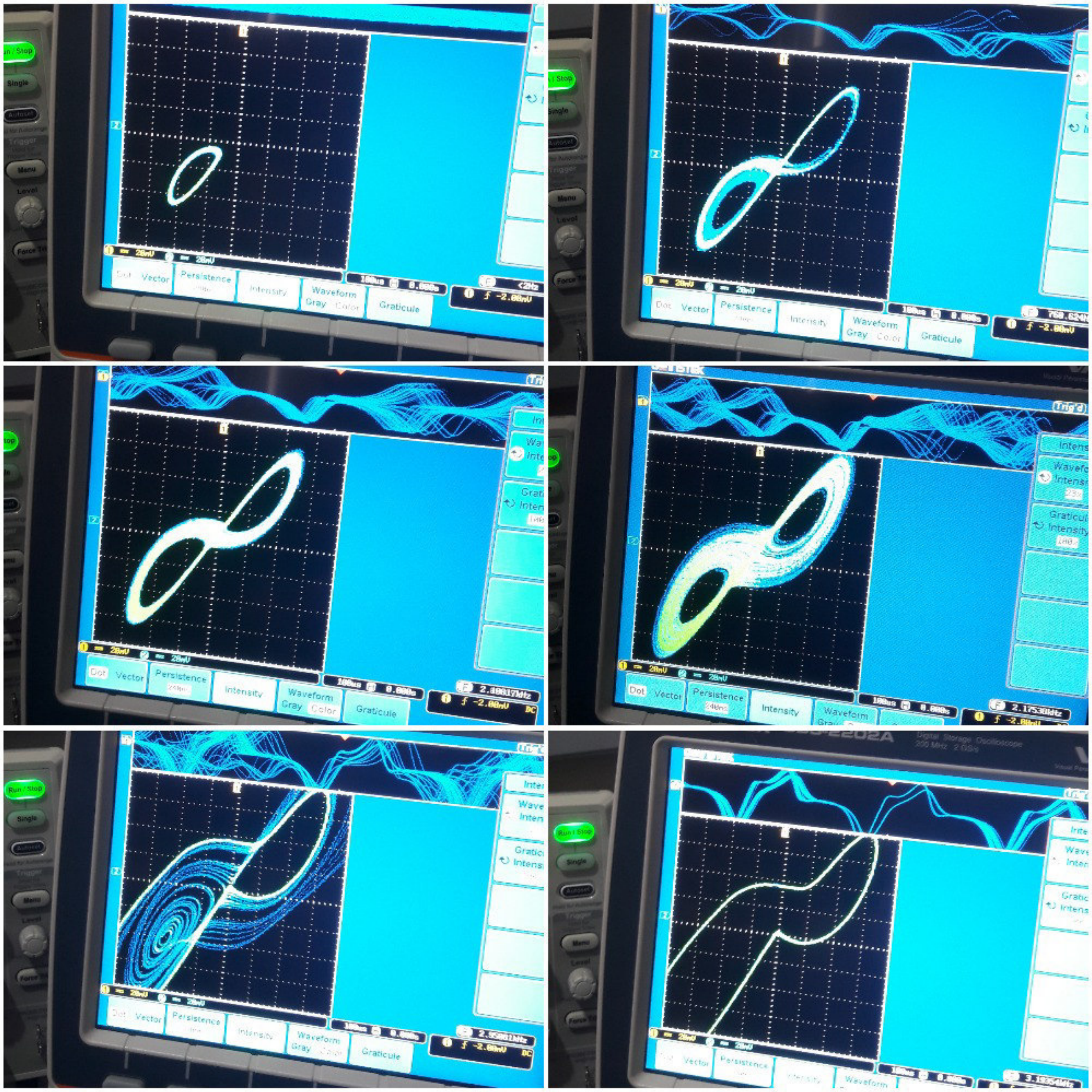


FIG. 11. Evolution of x vs y for varying R_8

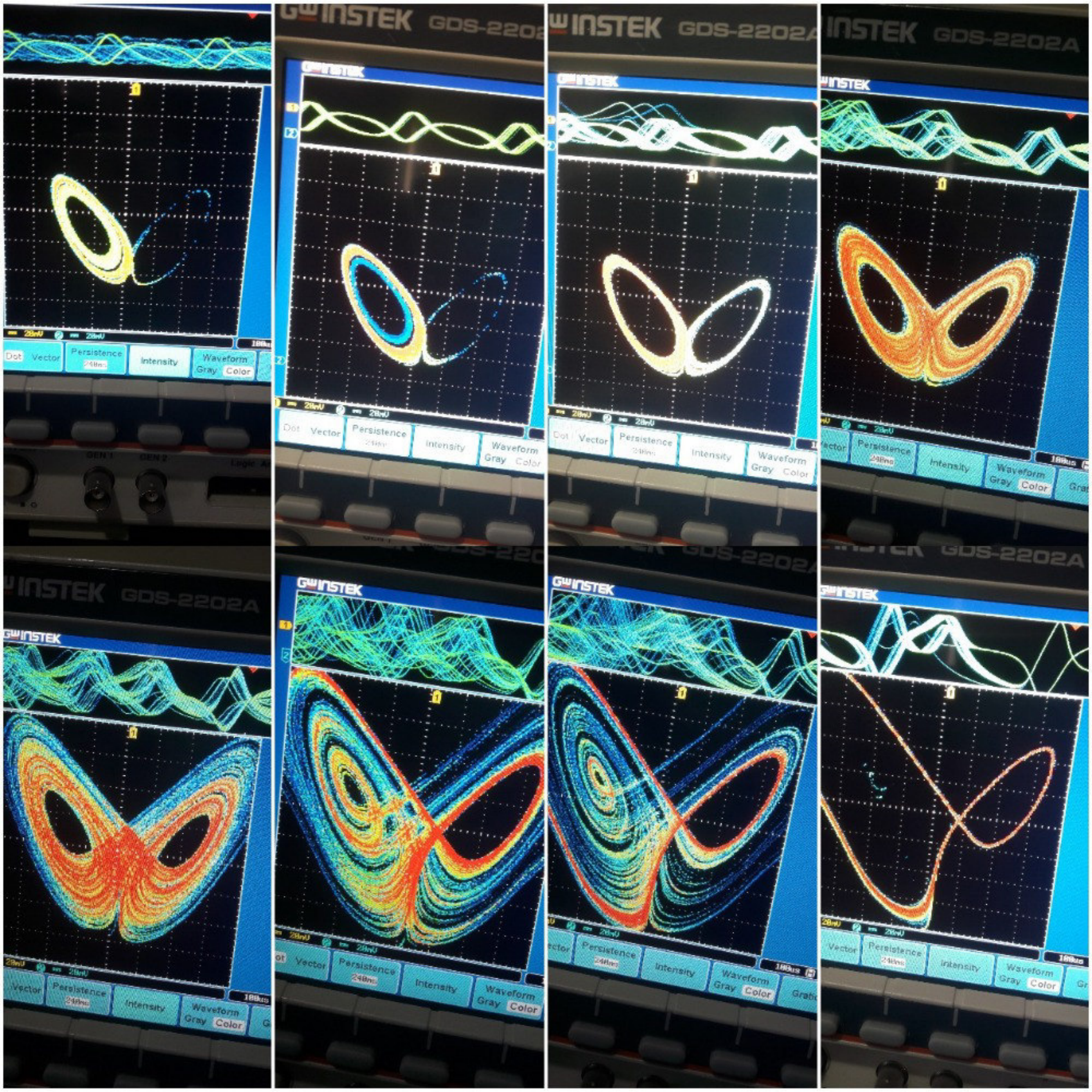


FIG. 12. Evolution of x vs z for varying R_8

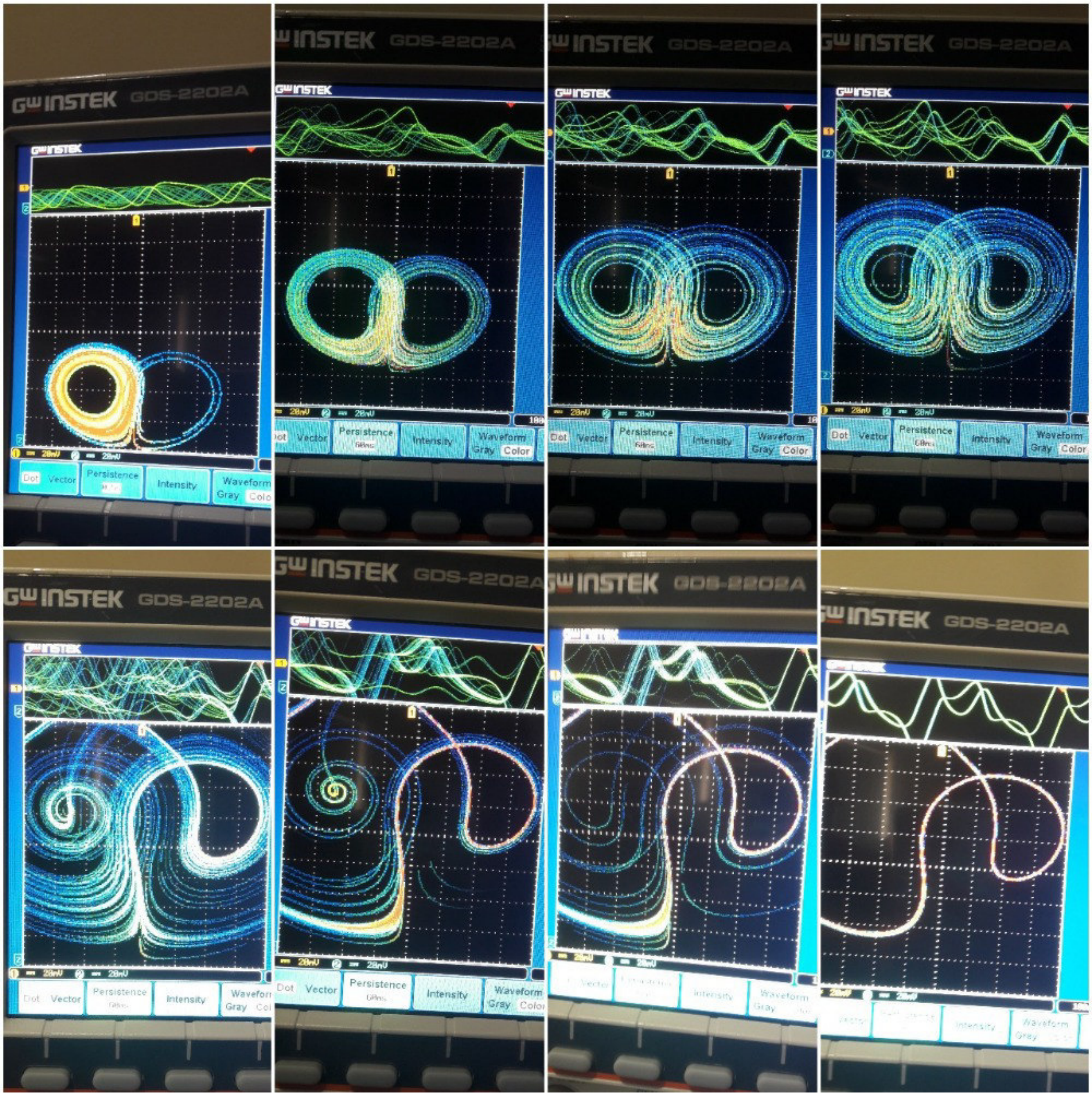


FIG. 13. Evolution of y vs z for varying R_8

Parameterization on R_{13} (β)

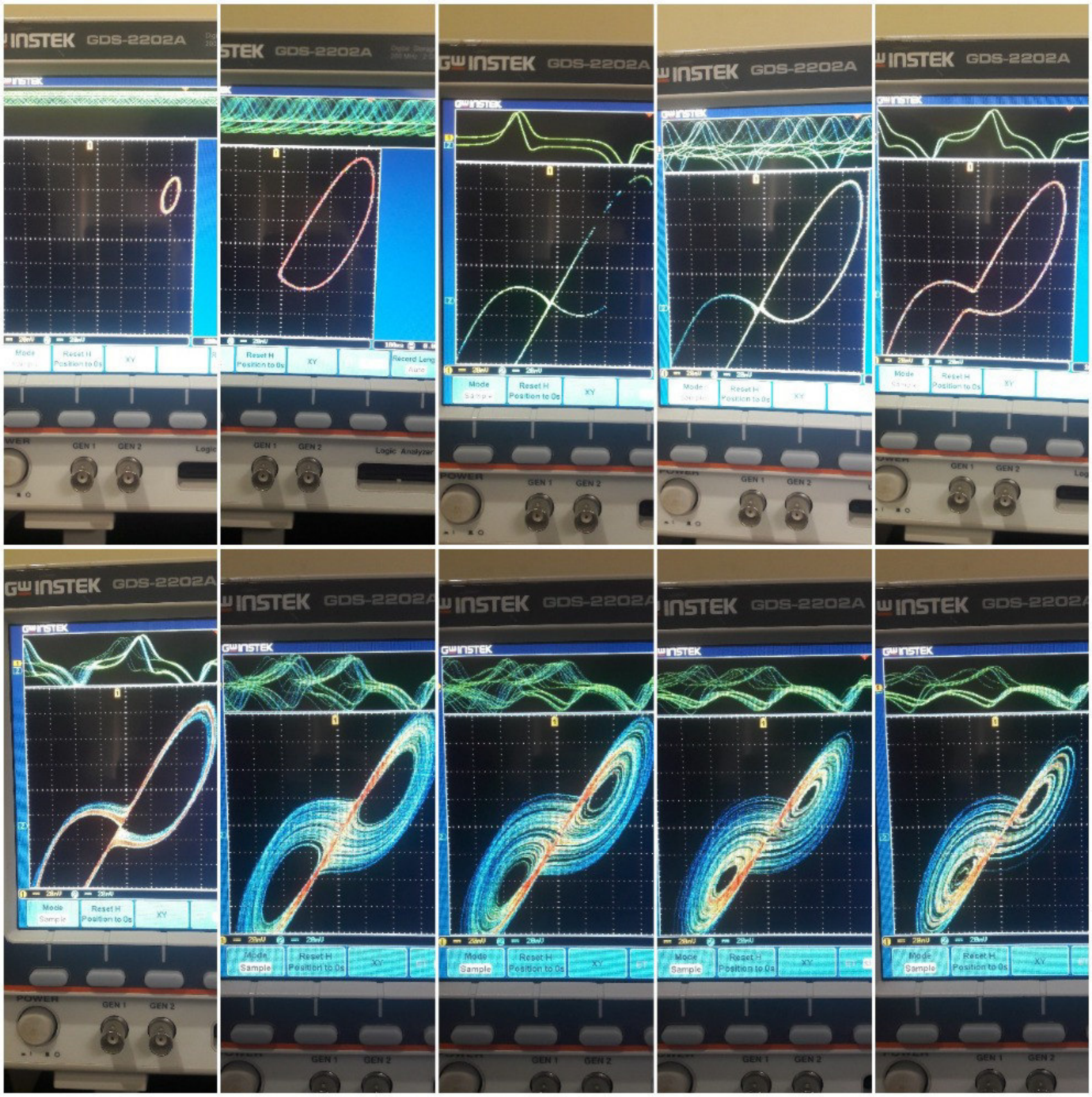


FIG. 14. Evolution of x vs y for varying R_{13}

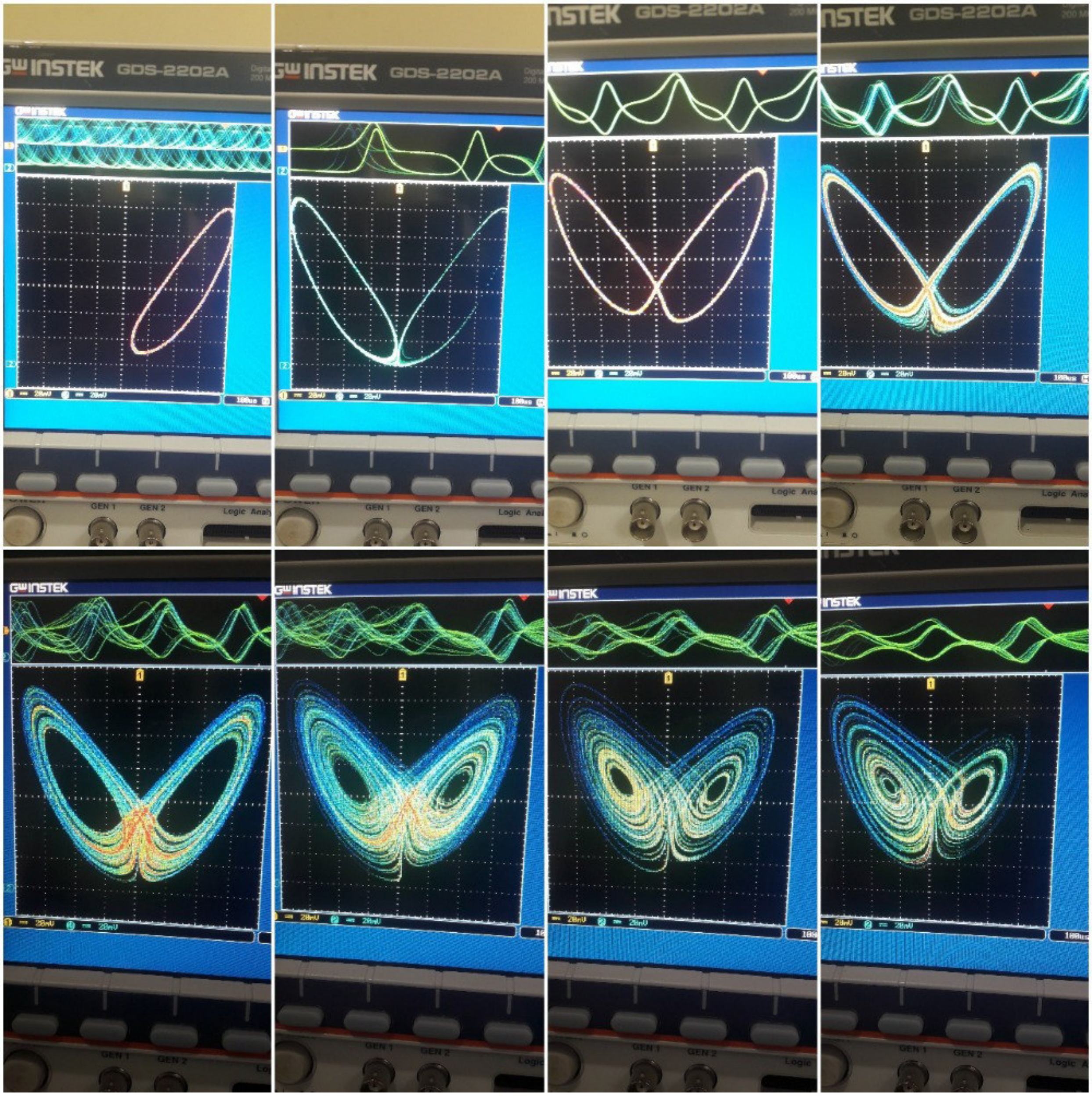


FIG. 15. Evolution of x vs z for varying R_{13}

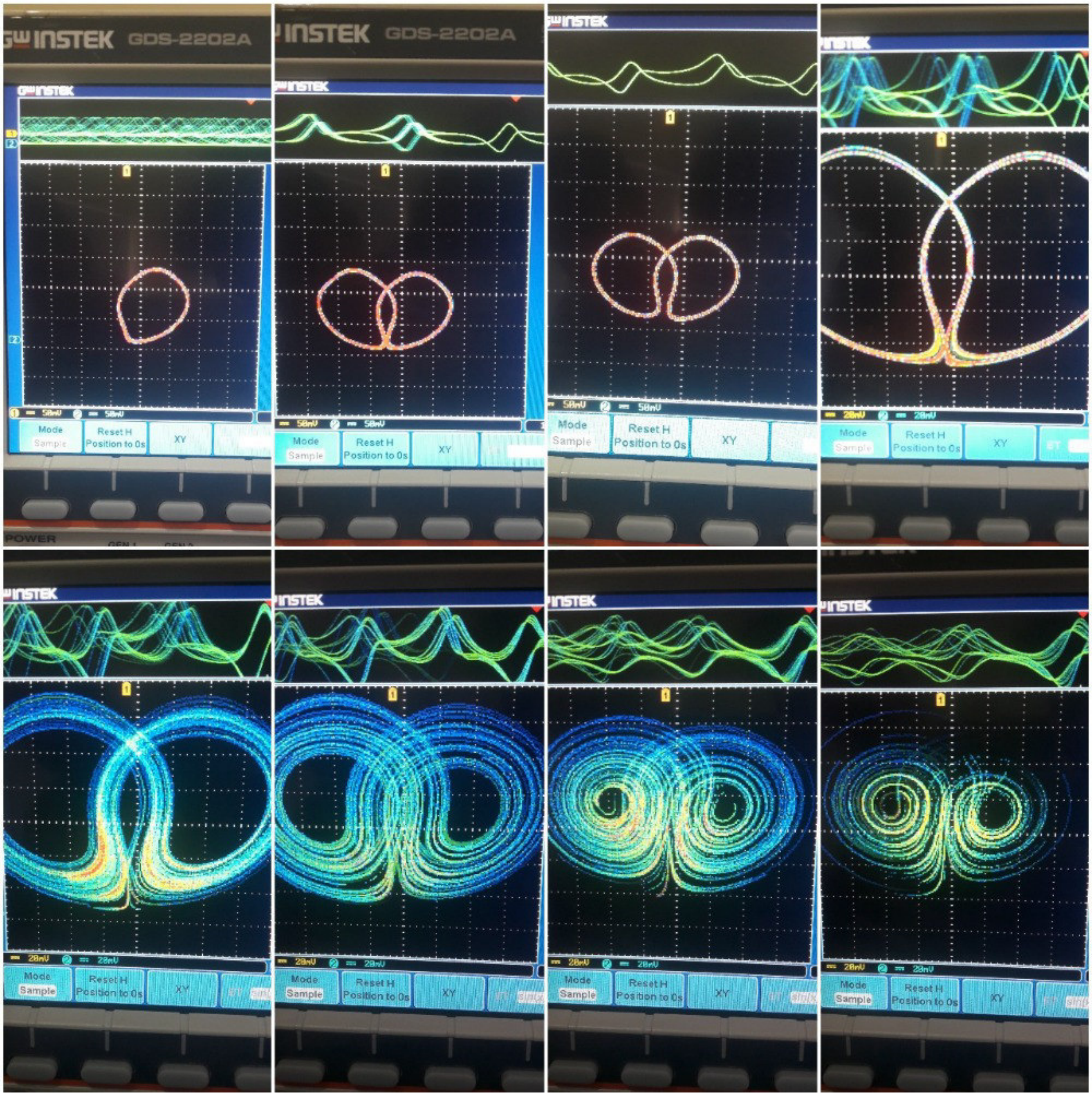


FIG. 16. Evolution of x vs z for varying R_{13}

The transition from single scroll to double scroll happens at $R_8 = 6.14\ k\Omega$ with other values held constant. And at $R_{13} = 15.88\ k\Omega$ with other values held constant.

SYNCHRONIZATION OF LORENZ-BASED CHAOTIC CIRCUITS

Introduction

Chaotic systems provide a rich mechanism for signal and generation, with potential applications to communications and signal processing. Because chaotic signals are typically broadband, noiselike, and difficult to predict, they can be used in various contexts for masking information-bearing waveforms.

Self-Synchronization property:

A chaotic system is self-synchronizing if it can be decomposed into at least two subsystems: a drive system (**transmitter**) and a stable response subsystem (**receiver**) that synchronize when coupled with a common signal.[4]
The Lorenz circuit is an example of **Synchronized chaotic systems (SCS)**. It's ability to synchronize is robust. Thus, in Lorenz system , the synchronization is highly robust to perturbations in the drive signal .

The process of Synchronization

First we need to construct two independent Lorenz Attractor circuits which have similar values of the parameters α, ρ and β. That is we need the circuit to have similar values of Resistances and Capacitors. Let the Transmitter Circuit be represented by these set of equations.

$$\dot{x} = \alpha(y - x)$$

$$\dot{y} = \rho x - xz - y$$

$$\dot{z} = xy - \beta z$$

Now , let us denote the receiver circuit by the set of equations

$$\dot{x_r} = \alpha(y_r - x_r)$$

$$\dot{y} = \rho x_r - x_r z_r - y_r$$

$$\dot{z} = x_r y_r - \beta z$$

To demonstrate the phenomenon of Synchronization, we replace x_r in the second and third equation of the receiver circuit with x of the transmitter circuit.
Thus, the receiver circuit is represented by

$$\dot{x} = \alpha(y - x)$$
$$\dot{y} = \rho x - x z_r - y_r$$
$$\dot{z} = x y_r - \beta z$$

This system is also refereed to as **x-drive** system. We denote the transmitter state variables collectively by the vector $\mathbf{d} = (x, y, z)$ and the receiver variable by the vector $\mathbf{r} = (x_r, y_r, z_r)$ when convenient.

Proof of Synchronization

By defining the dynamical errors by $\mathbf{e} = \mathbf{d} - \mathbf{r}$, it is straightforward to show that synchronization in the Lorenz system is a result of stable error dynamics between the transmitter and receiver. Assuming that the transmitter and receiver coefficients are identical, a set of equations that governs the error dynamics is given by

$$\dot{e_1} = \alpha(e_2 - e_1)$$
$$\dot{e_2} = -x(t)e_3 - e_2$$
$$\dot{e_3} = x(t)e_2 - \beta e_3$$

The error dynamics are globally asymptotically stable at the origin, provided that α, β > 0. This result follows by considering the 3D Lyapunov function defined by

$$E(e,t) = \frac{1}{2}(\frac{1}{\alpha}e_1^2 + e_2^2 + 4e_3^2)$$

Since E is positive definite and $\dot{E}$ is negative definite, this implies that $\mathbf{e}(t) \to 0$ as $t \to \infty$. Therefore, synchronization occurs as $t \to \infty$

Observations based on Synchronization

It is expected from the theory that x and x_r will follow similar pattern and so will the other variables y and z.
Thus, we expect that a plot of x_r vs x will be a straight line of slope 45^o. That is exactly what we observed.

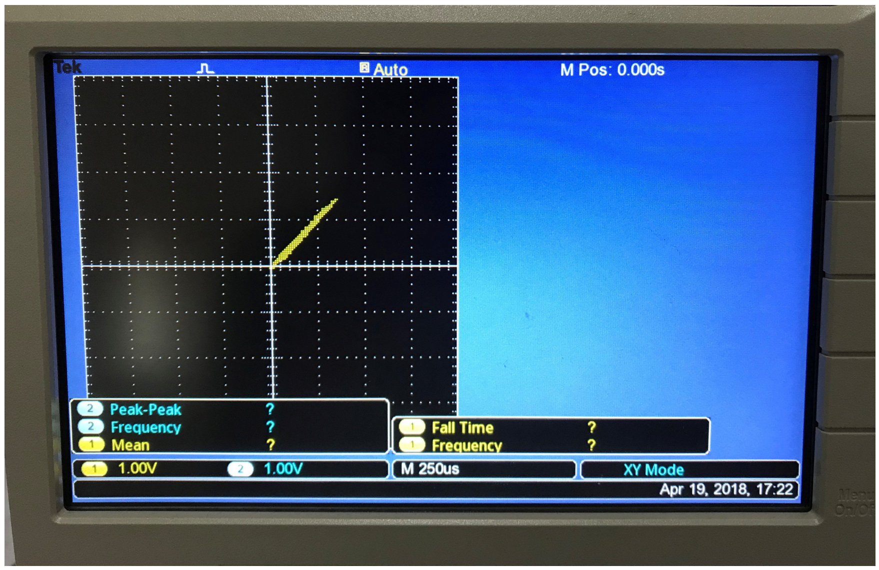


FIG. 17. Observed trend of x vs x_r in the oscilloscope.

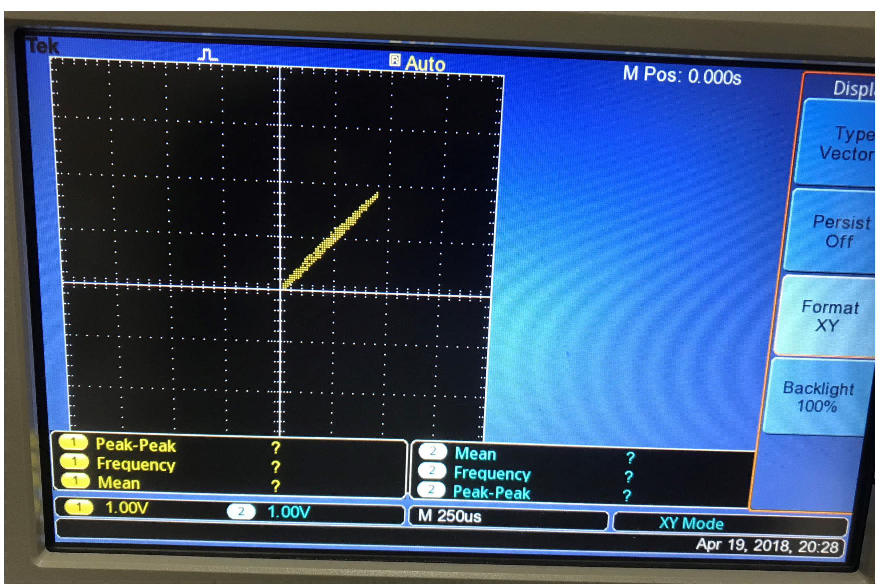


FIG. 18. Observed trend of y vs y_r in the oscilloscope.

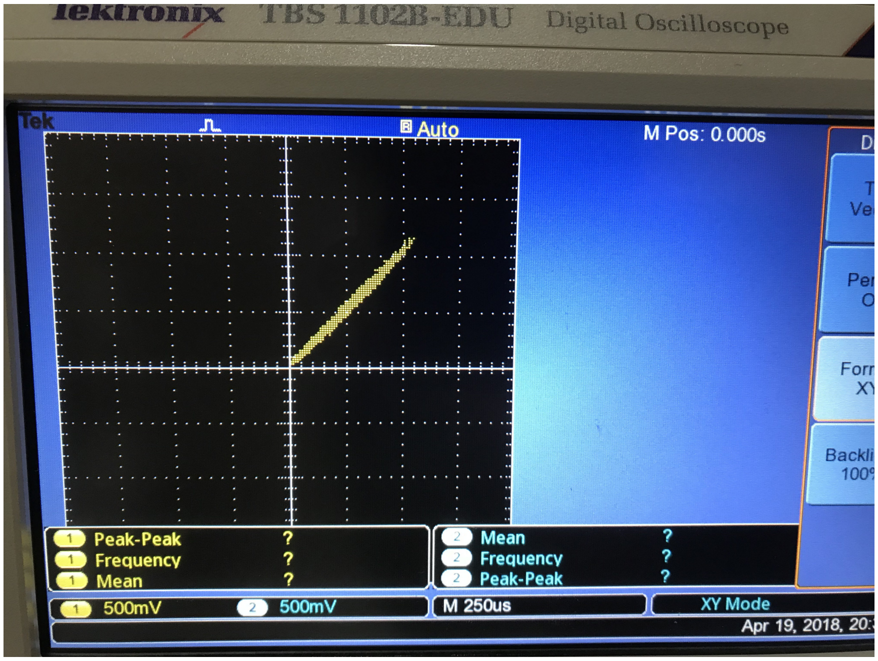


FIG. 19. Observed trend of z vs z_r in the oscilloscope.

Comparision of same variables from both receiver and transmitter circuit

As we can observe that $x^i(t)$ vs $x_r^i(t)$ (where $i = 1, 2, 3$) evolves in a similar fashion. Therefore synchronization is further verified.

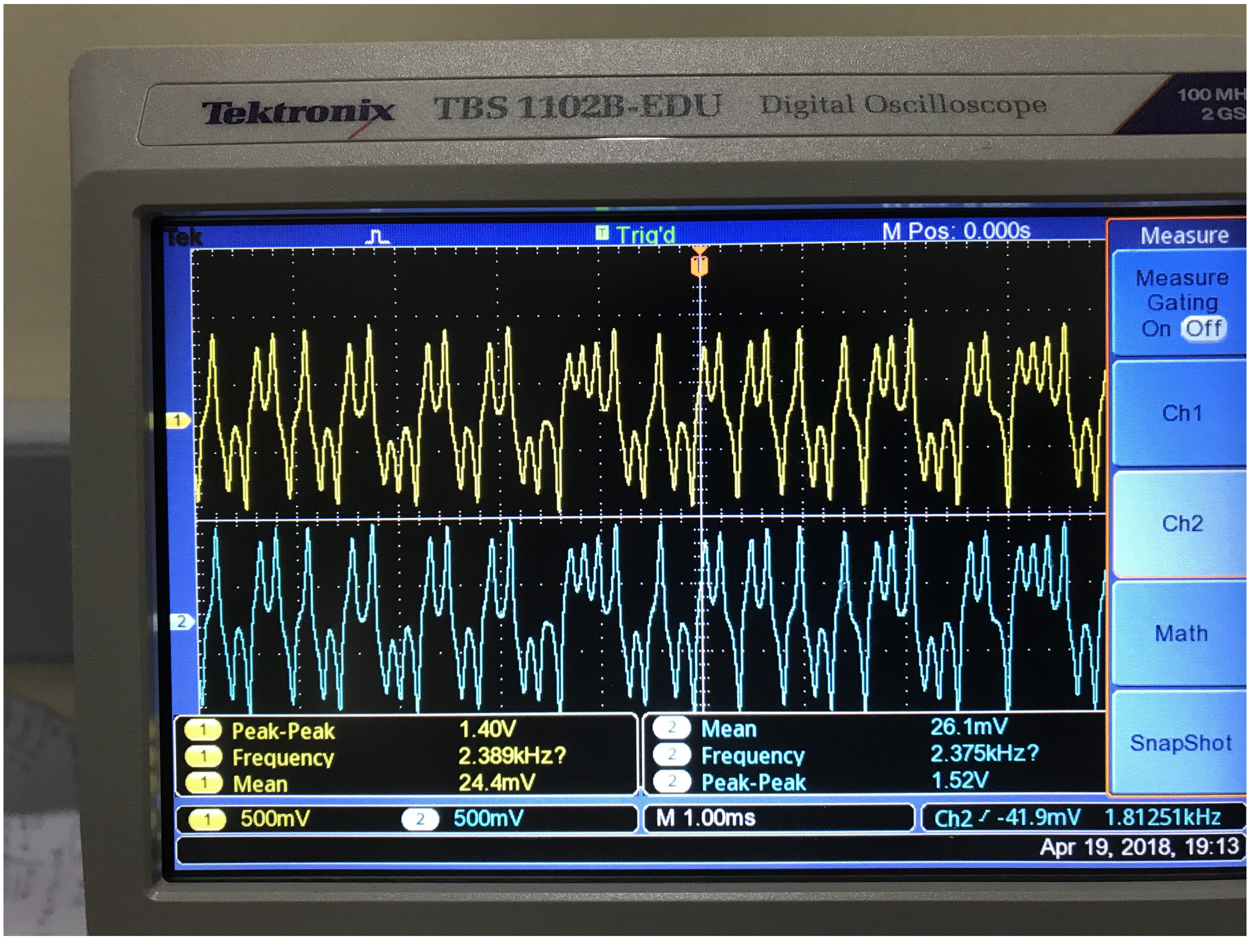


FIG. 20. Comparison of $x(t)$ vs $x_r(t)$.

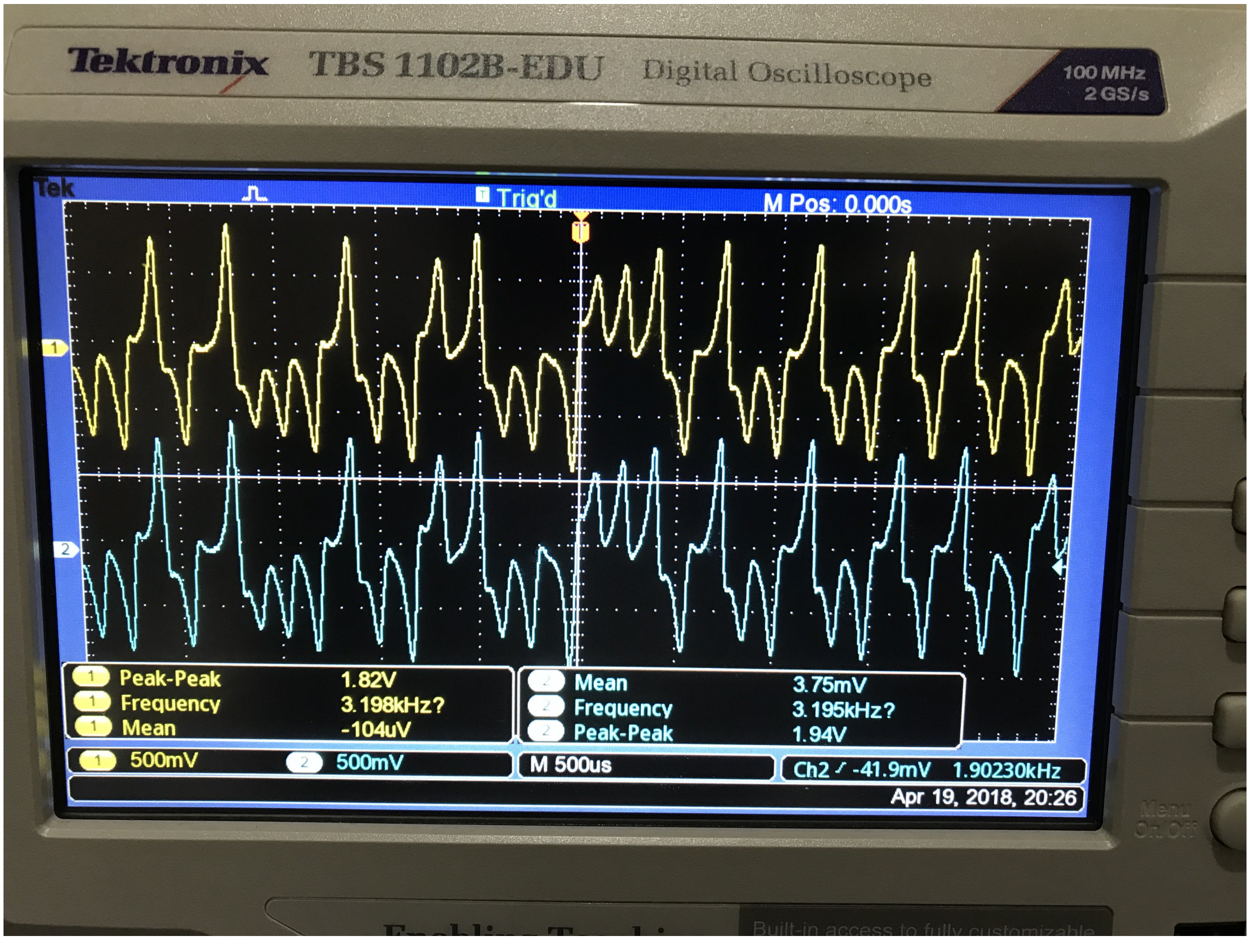


FIG. 21. Comparison of $y(t)$ vs $y_r(t)$.

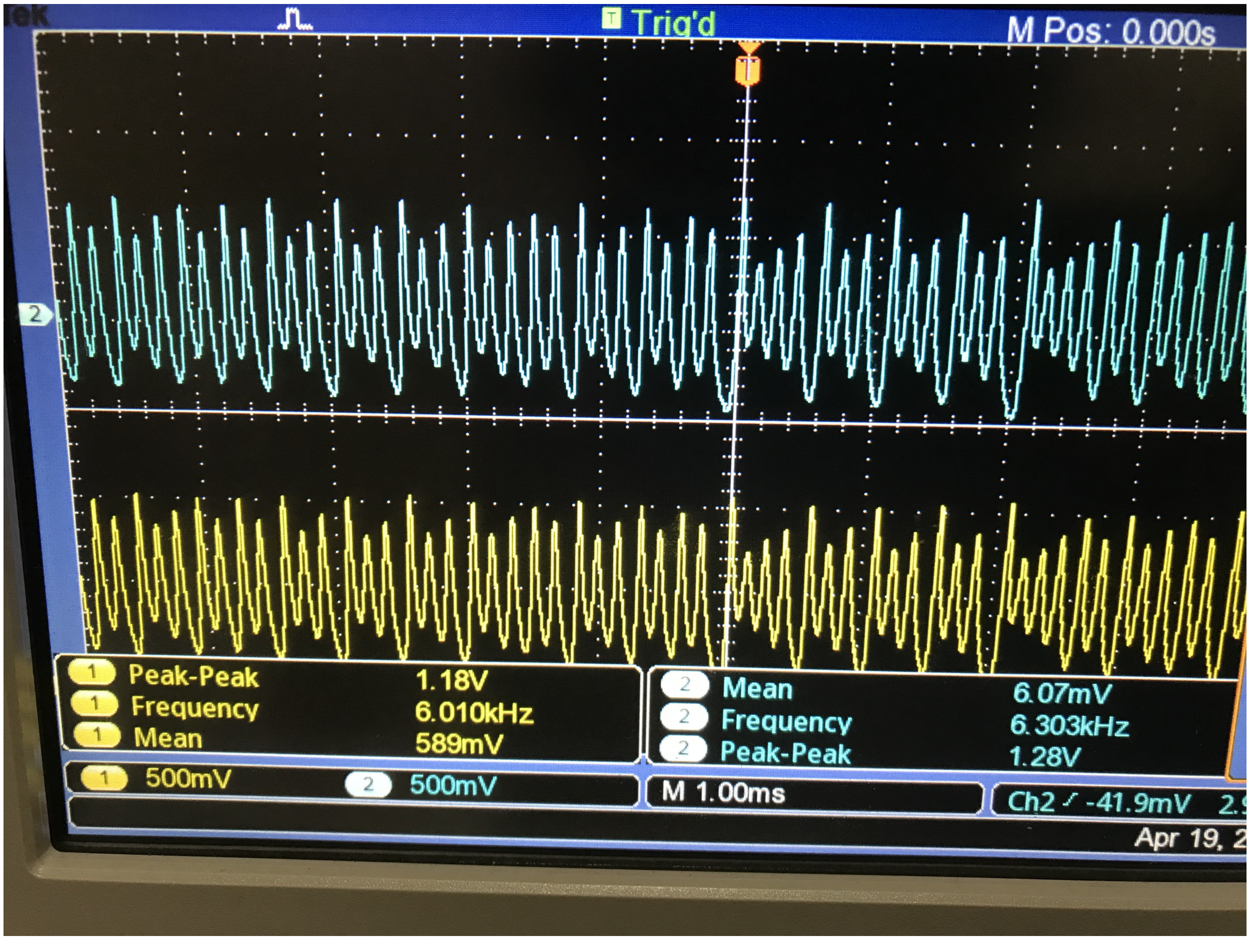


FIG. 22. Comparison of $z(t)$ vs $z_r(t)$.

Plots of cross terms

We get the typical Lorenz curve figures for xz, xy and zy with inputs coming from receiver and transmitter circuits respectively.

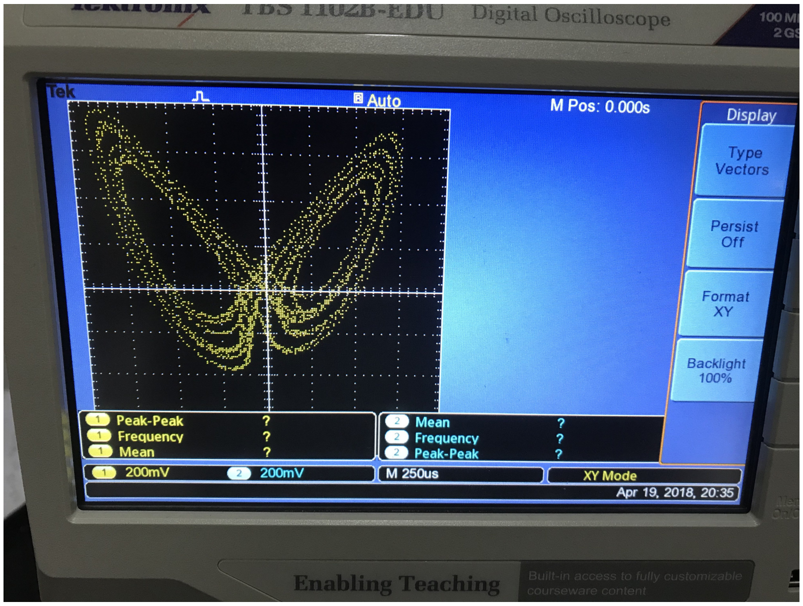


FIG. 23. $x(t)$ vs $z_r(t)$

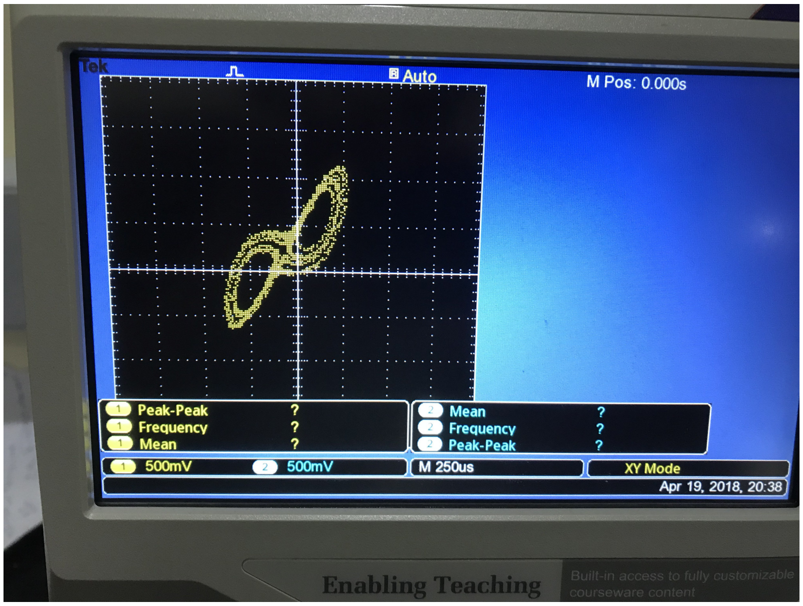


FIG. 24. $x(t)$ vs $y_r(t)$

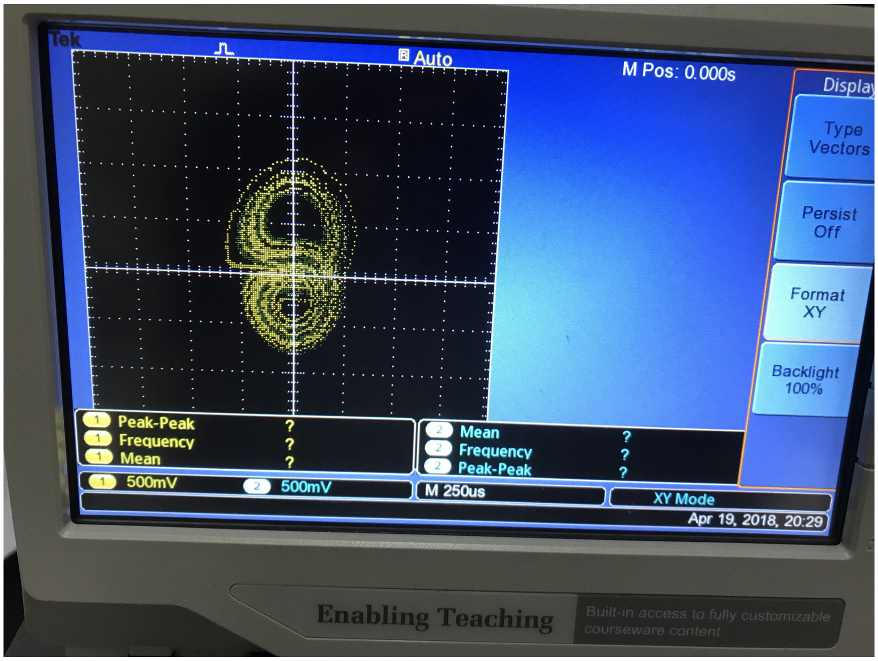


FIG. 25. $z(t)$ vs $y_r(t)$

Applications: Chaotic signal masking in secure Communication

A potential approach to communications applications is based on chaotic signal masking and recovery[3]. In signal masking, a noiselike masking signal is added at the transmitter to the information-bearing signal $m(t)$, and at the receiver the masking is removed. In our system, the basic idea is to use the received signal to regenerate the masking signal at the receiver and subtract it from the received signal to recover $m(t)$. This can be done with the synchronizing receiver circuit, since the ability to synchronize is found experimentally to be robust, i.e., is not highly sensitive to perturbations in the drive signal and thus can be done with the masked signal.

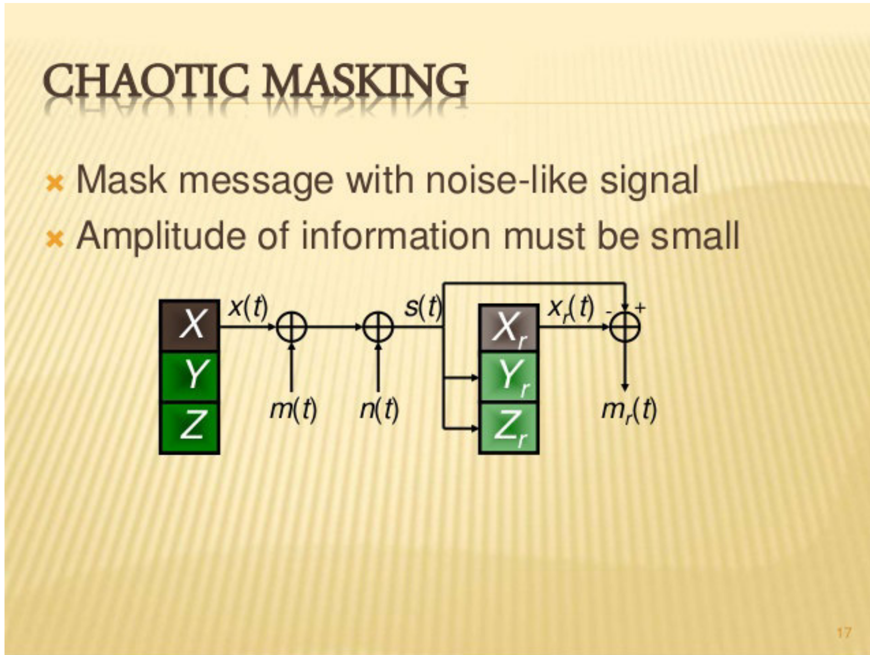


FIG. 26. A Chaotic Masking System

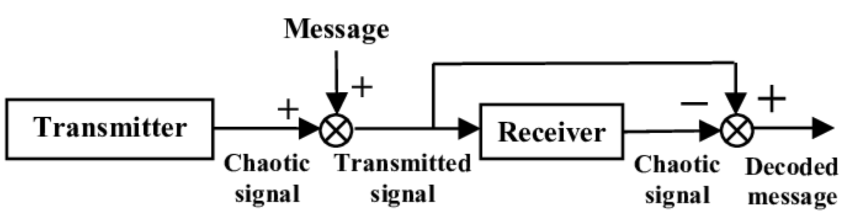
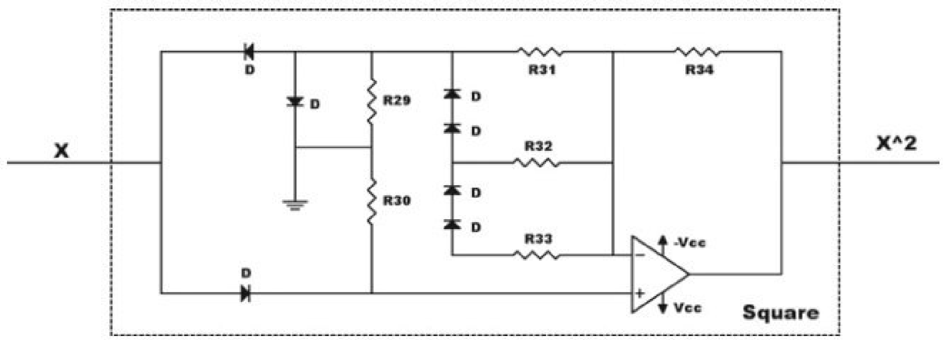


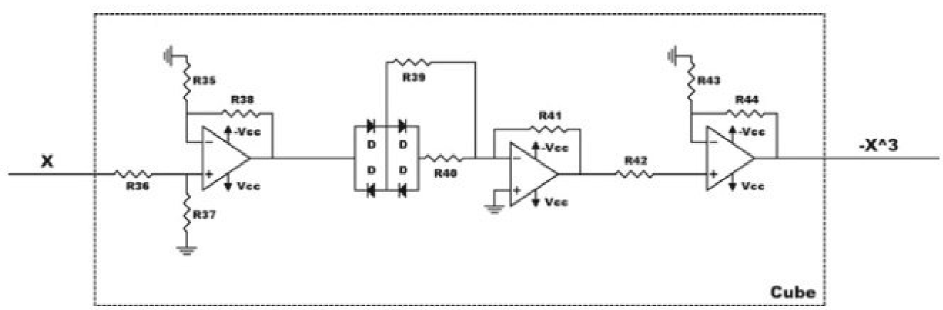
FIG. 27. A message is added between the transmitter and the receiver. The added message can then be retrieved by using a subtractor circuit at the output of the receiver.

PWL APPROXIMATION OF NONLINEARITIES

The nonlinear terms in a circuit can be approximated by using Piecewise linear approximation(PWL) of the nonlinear terms.[2] We constructed such PWL circuits for the implementation of x^2 and x^3 and found that the circuits functions in a fairly good manner upto certain voltage limits. They are useful and easy to implement as the circuitry implementing the PWL functions is realized with a few components like diodes and resistors. Thus, in some cases they can be used if there is a shortage of multipliers in the lab.



Circuital implementation of the square function. Components: $R_{29} = 10\,\mathrm{k\Omega}$, $R_{30} = 10\,\mathrm{k\Omega}$, $R_{31} = 10\,\mathrm{k\Omega}$, $R_{32} = 10\,\mathrm{k\Omega}$, $R_{33} = 4\,\mathrm{k\Omega}$, $R_{34} = 30\,\mathrm{k\Omega}$, 1$N$4148 Diode, $V_{cc} = 9\,\mathrm{V}$



Circuital implementation of the cube function. Components: $R_{35} = 200\,\mathrm{k\Omega}$, $R_{36} = 200\,\mathrm{k\Omega}$, $R_{37} = 100\,\mathrm{k\Omega}$, $R_{38} = 100\,\mathrm{k\Omega}$, $R_{39} = 12\,\mathrm{k\Omega}$, $R_{40} = 2\,\mathrm{k\Omega}$, $R_{41} = 15\,\mathrm{k\Omega}$, $R_{42} = 10\,\mathrm{k\Omega}$, $R_{43} = 10\,\mathrm{k\Omega}$, $R_{44} = 70\,\mathrm{k\Omega}$, 1$N$4148 Diode, $V_{cc} = 9\,\mathrm{V}$

FIG. 28. Circuit diagram for PWL approximation of x^2 and x^3

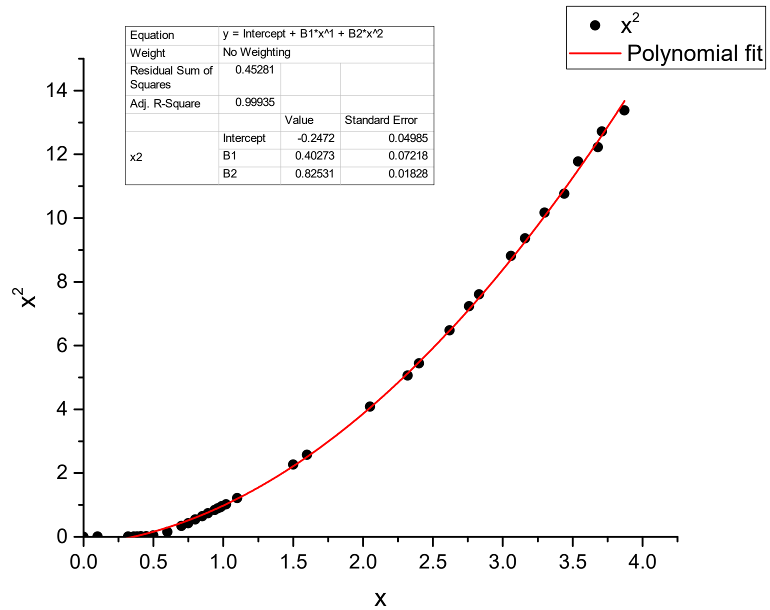


FIG. 29. Data obtained by PWL x^2 circuit. The data is fitted with a quadratic polynomial. It was observed to follow the expected trend to a good approximation.

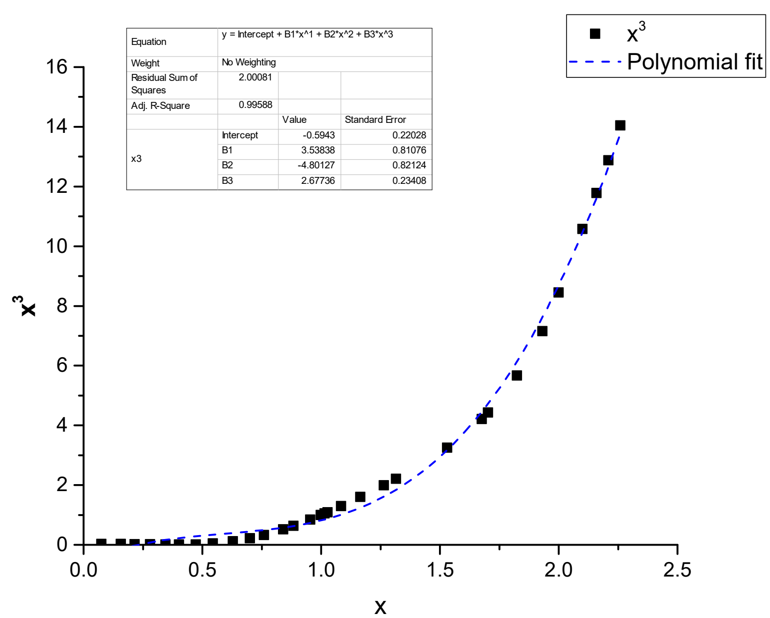


FIG. 30. Data obtained by PWL x^3 circuit. The data is fitted with a cubic polynomial. It was observed to follow the expected trend to a good approximation.

CONCLUSION

In our study and applications of Non-Linear Circuits, we have achieved the following things.

- Study of Lorenz Equations of 3D Chaotic Strange Attractor.
- Simulation of the Lorenz Differential Equations to achieve various plots.
- Implementation of Lorenz equations in electronic circuit using Op-Amps and Multipliers. Obtaining the corresponding signal in the oscilloscope and thus, matching with the simulated graphs.
- Parameterization based on changing of the variable resistors and observing the evolution of various plots of the Lorenz system.
- Study of self-synchronization property of Chaotic systems and verification of the same.
- Study of application of the synchronization property of the Lorenz system in Chaotic Signal Masking.
- Implementation of PWL circuits for the function x^2 and x^3

FURTHER

We could not get a suitable signal source $m(t)$ (Amplitude $<<$ 1 V) to use for our Chaotic Masking system. This is something that can be worked further upon to make a system of Lorenz circuits for secure communication.

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